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Topological string in harmonic space and correlation functions in S^3 stringy cosmology

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Abstract

We develop the harmonic space method for conifold and use it to study local complex deformations of T^*S^3 preserving manifestly $SL(2,C)$ isometry. We derive the perturbative manifestly $SL(2,C)$ invariant partition function $\mathcal{Z}_{top}$ of topological string B model on locally deformed conifold. Generic n momentum and winding modes of $2D$ $c=1$ non critical theory are described by highest $\upsilon_{(n,0)}$ and lowest components $\upsilon_{(0,n)}$ of $SL(2,C)$ spin $s=\frac{n}{2}$ multiplets $\left(\upsilon_{(n-k,k)}\right)$, $0 \leq k \leq n$ and are shown to be naturally captured by harmonic monomials. Isodoublets ($n=1$) describe uncoupled units of momentum and winding modes and are exactly realized as the $SL(2,C)$ harmonic variables U_α^+ and V_α^-. We also derive a dictionary giving the passage from Laurent (Fourier) analysis on T^*S^1 (S^1) to the harmonic method on T^*S^3 (S^3). The manifestly $SU(2,C)$ covariant correlation functions of the S^3 quantum cosmology model of Gukov-Saraikin-Vafa are also studied.

Keywords:

harmonic analysis on conifold, topological string B model on T^*S^3, ground ring of 2D $c=1$ string, Hartle-Hawking wave function and $\mathbb{S}^3$ quantum cosmology.

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1 Introduction

In the few last years, there has been a growing interest in the study of the interface between 10D superstring theory and cosmology [1]-[5]. This particular interest has been propelled more by recent developments in stringy cosmology; in particular by: (**1**) the remarkable observation on the link between Hartle-Hawking (HH) wave $\Psi = \Psi\left(\mathrm{t}, \overline{\mathrm{t}}\right)$ and the partition function $\mathcal{Z}_{top} = \mathcal{Z}_{top}\left(\mathrm{t}\right)$ of the topological string B model on the conifold $T^{*}S^{3}$ [6]-[14]. (**2**) The intimate relation of $\mathcal{Z}_{top}$ to the partition function $\mathcal{Z}_{c=1}$ of the $c = 1$ non critical bosonic string propagating on a circle at the self-dual radius [15]-[18]. (**3**) the interpretation of $\mathcal{Z}_{top}$ as an exact "wave function of the universe" in the mini-superspace sector of superstring theory [19]-[20]. These correspondences have allowed moreover to shed light on the link between the entropy of black holes and topological string B-model partition function [21]-[29]. They have also been used in [30] for the set up of Gukov-Saraikin-Vafa (GSV) S^{3} stringy cosmology model.

The aim of this paper is to complete partial results on $\mathcal{Z}_{top}\left(\mathrm{t}\right)$ and $\Psi\left(\mathrm{t}, \overline{\mathrm{t}}\right)$ by using the harmonic space method. Our contribution involves the following three: First we develop a manifestly covariant $SL\left(2, C\right)$ harmonic analysis to approach conifold geometry and it submanifolds with isometries $SL\left(2, C\right)/C^{*}$, $SU\left(2, C\right)$ and $SU\left(2, C\right)/U\left(1\right)$. Second we use this covariant formalism to study the $SL\left(2, C\right)$ invariant partition function of B model topological string theory on conifold. Third we construct the manifestly $SL\left(2, C\right)$ covariant correlation functions of the scaling field of stringy wave function for S^{3} quantum cosmology of Gukov-Saraikin-Vafa.

To make a general idea on the content of this paper, recall that, roughly, $\mathcal{Z}_{top}$ is a function of the moduli space of conifold local complex deformations $\left\{\mathrm{t}_{+n}, \widetilde{\mathrm{t}}_{-n}, ...\right\}$, $n \geq 0$. The latters are generated by an analytic function $\mathrm{t} = \mathrm{t}\left(x, y, z, x\right)$ living on $T^{*}S^{3}$ [31, 32]; and so $\mathcal{Z}_{top}$ can be thought of as a functional living on complex three dimensional conifold; $\mathcal{Z}_{top} = \mathcal{Z}_{top}\left[T^{*}S^{3}\right]$. Moreover like any function F on this manifold , $\mathcal{Z}_{top}$ should sit in a reducible representation of the $SL\left(2, C\right)$ isometry of $T^{*}S^{3}$. However, though the $SL\left(2, C\right)$ representation group structure of $\mathcal{Z}_{top}\left(T^{*}S^{3}\right)$ is quite known, its explicit expression with manifest $SL\left(2, C\right)$ symmetry is still missing. Notice by the way that explicit relations with manifest $SL\left(2, C\right)$ isometry of $\mathcal{Z}_{top}\left(T^{*}S^{3}\right)$ are important in the study of higher order corrections in complex moduli $\left\{\mathrm{t}_{+n}, \widetilde{\mathrm{t}}_{+n}, ...\right\}$; they are equally useful for the study of the couplings of states involving both momentum and winding modes. Recall also that when looking for explicit results, the set of local complex deformations $\mathrm{t} = \mathrm{t}\left(x, y, z, w\right)$ of the conifold $xy - zw = \mu$ is in general restricted to:

(**a**) the particular subset of local complex deformations $\mathrm{t}| = \mathrm{t}\left(x, y\right)| = \mathrm{t}\left(x, y, 0, 0\right)$ generating a special class of conifold local complex deformations. More precisely, perturbation by $\mathrm{t}(x,y)|$ deals with the local deformations of the complex one dimension submanifold $(xy - zw = \mu)|_{z=w=0} = \mu$. Going beyond this technical restriction, would be then an interesting task.

(**b**) using an approximation in which local complex deformations are split as $\mathrm{t}(x,y)| = \tau(x) + \widetilde{\tau}(y)$ with Laurent expansion

$$\tau(x) = \sum_{n\geq 1} \mathrm{t}_n x^n, \qquad \widetilde{\tau}(y) = \sum_{m\geq 1} \mathrm{t}_{-m} y^m. \tag{1.1}$$

In this approximation, couplings type $\tau(x) \times \widetilde{\tau}(y)$ and higher orders in $\tau(x)$ and $\widetilde{\tau}(y)$ are treated as small fluctuations and so ignored. This procedure is a priori valid provided that the conditions $\tau(x) << \mu$ and $\widetilde{\tau}(y) << \mu$ are fulfilled. Under this approximation, couplings type

$$t_n t_{-m} x^n y^m \tag{1.2}$$

involving both x and y mode variables are then treated as higher order corrections $(\tau(x)\widetilde{\tau}(y) \simeq \mathcal{O}(2))$. Though this method allows useful simplifications; there is however a price to pay; it is a selective method since only a part of the excitation modes associated with the deformation monomials $x^n y^m$, $x^n z^m$, $x^n w^m$, $y^n z^m$, $y^n w^m$ and $z^n w^m$ with $k = m + n$, that is

$$\begin{array}{ccc} \sum\limits_{m+n=k} t_n t_{-m} x^n y^m, & \sum\limits_{m+n=k} t_n s_m x^n z^m, & \sum\limits_{m+n=k} t_n s_{-m} x^n w^m, \\ \sum\limits_{m+n=k} t_{-n} s_{-m} y^n z^m, & \sum\limits_{m+n=k} t_{-n} s_{-m} y^n w^m, & \sum\limits_{m+n=k} s_n s_{-m} z^n w^m, \end{array}$$

is considered. From 2D $c = 1$ string view, this approximation disregards not only winding modes $(s_n z^n + s_{-m} w^m)$ and their interactions $s_n s_{-m} z^n w^m$; but ignores as well the contributions coming from couplings between the $t_{\pm n}$-$s_{\pm m}$ mode components of the $SL(2,C)$ spin $j = \frac{k}{2}$ multiplet. Notice that the omitted zero mode $n = m = 0$ in the expansion of $\tau(x)$ and $\widetilde{\tau}(y)$ corresponds just to the usual global deformation parameter μ $(t_0 = \mu)$; while positivity of integers n and m is required by their interpretations as highest weights of $SL(2,C)$ representations.

Motivated by these partial results, we address the question of building a manifestly $SL(2,C)$ harmonic formalism for the B model topological string partition function $\mathcal{Z}_{top}[T^*S^3]$ and the HH wave function $\Psi[S^3]$ of GSK model. With this harmonic formalism at hand, one is able to study local complex deformations of conifold and complete the literature partial results. We show in particular that conifold local complex deformations and manifestly $SL(2,C)$ topological string computations, involving both momenta and winding modes as well as the $SU(2,C)$ covariant analysis of the correlation functions of GSK quantum cosmology model get a natural formulation in harmonic space language.

Among our results, we mention the three following: In the perturbative complex deformation approach, the locally deformed conifold is given by

$$u^{+\alpha}v_{\alpha}^{-}=\mu+\xi\left(u^{+},v^{-}\right).$$

The local complex deformations, captured by the harmonic function $\xi\left(u^{+},v^{-}\right)$, can be expressed in terms of the irreducible modes $\zeta^{\pm n}$ living on $T^{*}S^{2}$ with harmonic expansions,

$$\zeta^{+n}=\zeta_{(\alpha_{1}...\alpha_{n})}u^{+\alpha_{1}}...u^{+\alpha_{n}},\qquad \widetilde{\zeta}^{-n}=\widetilde{\zeta}^{(\beta_{1}...\beta_{n})}v_{\beta_{1}}^{-}...v_{\beta_{n}}^{-}\quad .$$

They generate special infinitesimal local complex deformations using the harmonic variables $u^{+\alpha}$ and v_{α}^{-} and the $SL\left(2\right)$ tensors $\zeta_{(\alpha_{1}...\alpha_{n})}$ and $\widetilde{\zeta}^{(\beta_{1}...\beta_{n})}$. The use of the harmonic sections $\zeta^{\pm n}$ allows to avoid the technical difficulties one encounters in the standard complex analysis and permits to go beyond the $T^{*}S^{1}$ restriction currently used in literature.

For the B model topological string on conifold, the harmonic space partition function preserving manifestly $SL\left(2,C\right)$ symmetry factorizes as usual as $\mathcal{Z}_{top}\left(\zeta^{n},\widetilde{\zeta}^{m}\right)=\exp\left[\mathcal{F}\left(\zeta^{n},\widetilde{\zeta}^{m}\right)\right]$ with free energy

$$\mathcal{F}=\sum_{g=0}^{\infty}\left(\frac{g_{s}}{\mu}\right)^{2g-2}\mathcal{F}_{g}$$

but instead of Laurent modes, we have now the harmonic sections $\zeta^{\pm n}$. For instance, we find that the leading terms of the genus zero free energy read as,

$$\begin{aligned}\mathcal{F}_{0}\;&=\;-\frac{1}{g_{s}^{2}}\sum_{n>0}\frac{\mu^{n-2}}{n}\int_{T^{*}S^{2}}\left(\zeta^{n}\widetilde{\zeta}^{-n}\right)\\&+\frac{1}{g_{s}^{2}}\sum_{n_{1}+n_{2}+n_{3}=0}\mu^{\frac{|n_{1}|+|n_{2}|+|n_{3}|-2}{2}-2}\left(\int_{T^{*}S^{2}}\zeta^{n_{1}}\zeta^{n_{2}}\zeta^{n_{3}}\right)+...\quad .\end{aligned}\tag{1.3}$$

In this relation, complex 3D conifold is defined as $u^{+\alpha}v_{\alpha}^{-}=\mu$, with global complex parameter μ. Restriction to S^{3} is recovered by setting $v_{\alpha}^{-}=u_{\alpha}^{-}$ and $p=\mathrm{Re}\,\mu$.

For GSK quantum cosmology model on S^{3}, we find that N points correlation functions $G_{N}=<\Phi\left(U_{1}^{\pm}\right)...\Phi\left(U_{N}^{\pm}\right)>$ of the conformal field factor Φ on the 3-sphere read as,

$$G_{N}=\mathcal{N}\int D\zeta D\overline{\zeta}\left(\prod_{i=1}^{N}\Phi_{i}\left(U_{i}^{\pm}\right)\right)\exp\left(-\frac{1}{g_{s}^{2}}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]\right)\quad ,\tag{1.4}$$

with

$$\begin{aligned}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]\;&=\;\int_{S^{3}}\zeta\frac{1}{D^{--}D^{++}+D^{++}D^{--}}\overline{\zeta}\\&-\frac{1}{6p}\int_{S^{3}}\left(\zeta^{2}\overline{\zeta}+\zeta\overline{\zeta}^{2}\right)+\mathcal{O}\left(\frac{1}{p^{2}}\left(\zeta\overline{\zeta}\right)^{2},...\right)\quad ,\end{aligned}\tag{1.5}$$

where $D^{--} = u^- \frac{\partial}{\partial u^+}$ and $D^{++} = u^+ \frac{\partial}{\partial u^-}$ are harmonic derivatives to be given later. As we see, our harmonic analysis permits to go beyond the standard computations restricted to the T^*S^1 conifold subvariety and also beyond the usual analysis restricted to the large circle $x\overline{x} = \mathrm{Re}\,\mu$ (with $z = 0$) of S^3 parameterized by $x\overline{x} + z\overline{z} = \mathrm{Re}\,\mu$.

The presentation of this work is as follows: In section 2, we make a preliminary discussion by anticipating on some of the results of this paper; this may help to make an idea on the method we will be using and the way we want to do things. In section 3, we recall useful results on the ground ring of the $c = 1$ string theory and conformal deformations. In section 4, we introduce harmonic analysis to study complex deformations of the conifold by putting in front group theoretic properties of its $SL(2, C)$ isometry group. The properties of the real slice of T^*S^3 are recovered by imposing unimodularity condition. In section 5, we give the classification of its complex deformations by using the harmonic frame work and establish a dictionary giving correspondence between Fourier method on the circle and harmonic analysis on 3-sphere. In section 6, we study the partition function $\mathcal{Z}_{top}$ of B-model topological string on T^*S^3 by using harmonic frame work and derives its manifestly $SL(2, C)$ expression. In section 7, we consider the S^3 quantum cosmology model within the harmonic coordinate set up. We study the manifestly $SU(2, C)$ invariant expression of Hartle-Hawking wave function in the S^3 cosmology model of GSV. Then, we compute the correlation functions of the conformal field on the full sphere S^3. Section 8 is devoted to conclusion and in section 9 we give an appendix on technical details regarding harmonic analysis on conifold and its submanifolds T^*S^2 , S^3 and S^2.

2 Preliminaries and overview

Before going into technical details, we think it is instructive to anticipate this study by summarizing briefly the main results obtained in this paper. This allows to fix the ideas on the harmonic variables method and the way we will handle $\mathcal{Z}_{top}$ of B model topological string on conifold and Hartlee-Hawking wave function on 3-sphere.

First of all, recall that harmonic analysis on the 2-sphere S^2 has been used successfully in the past, in particular in the study of the manifestly off shell formulation of $4D$ $\mathcal{N} = 2$ extended supersymmetry [33]-[37]; $4D$ $\mathcal{N} = 2$ supergravity [38]-[41] and in the building of $4D$ Euclidean Yang Mills and gravitational instantons [42]-[48]. Here, we shall go beyond this formalism since we shall deal with harmonic analysis on conifold T^*S^3 and its connection with the quantum ground ring

$$\mathcal{V} = \mathcal{R} \otimes \overline{\mathcal{R}} \tag{2.1}$$

of 2D $c=1$ string theory [16, 15]; see also [49]-[53]. The above mentioned Galperin *et al* harmonic analysis on S^2 appears in our construction as a particular case. The point is that here we are considering a large space namely T^*S^3; its compact real slice S^3 is related to the 2-sphere through the usual fibration $S^1 \times S^2$. Let us give below an overview on these tools; more rigorous details will be given in forthcoming sections.

2.1 Harmonic variables and 2D $c = 1$ string

2.1.1 Harmonic variables

To approach $\mathcal{Z}_{top}(T^*S^3)$ and, upon imposing reality condition, the Hartlee-Hawking wave function $\Psi(S^3)$ we shall use the following two pairs of harmonic variables

$$\left(U_\alpha^+, U_\alpha^-\right), \qquad \alpha = 1,2 \quad , \tag{2.2}$$

and

$$\left(V_\beta^+, V_\beta^-\right), \qquad \beta = 1,2 \quad . \tag{2.3}$$

Each pair transforms as an isodoublet under $SU(2)$ and parameterizes two separate complex spaces $C^2 \sim R^4$. To fix the ideas, we shall sometimes supplement the quantities living the u-space by an extra sub-index and the same for the v-space quantities. For instance C_u^2 refers to the complex space parameterized by eq(2.2) and $SU_u(2)$ the isometry group $SU(2)$ rotating the $U_\alpha^\pm$ variables.
The $U_\alpha^+ = \left(U_1^+, U_2^+\right)$ and $V_\beta^- = \left(V_1^-, V_2^-\right)$ are four complex holomorphic variables parameterizing C^4; the $U_\alpha^- = \overline{U^{+\alpha}}$ and $V^{+\beta} = \overline{V_\beta^-}$ are their complex conjugate. Viewed collectively, these variables are subject to the homogeneous constraint eq

$$\begin{array}{ccccc} U_\alpha^+ & \longrightarrow & \lambda U_\alpha^+ & & , \\ V_\beta^- & \longrightarrow & \frac{1}{\lambda} V_\beta^- , & \lambda \in C^* & , \end{array} \tag{2.4}$$

so that functions generated by monomials type

$$\prod_{i=0}^{n} U_{\alpha_i}^+ V_{\beta_i}^- \tag{2.5}$$

are invariant under C^* transformations. A simple example is given by the projective complex three dimension surface

$$\varepsilon^{\alpha\beta} U_\alpha^+ V_\beta^- = \mu, \tag{2.6}$$

where μ is complex constant. Viewed separately, the $U_\alpha^\pm$ variable are restricted as

$$\begin{array}{ccccc} U_\alpha^+ & \longrightarrow & e^{i\theta} U_\alpha^+ & & , \\ U_\alpha^- & \longrightarrow & e^{-i\theta} U_\alpha^- , & \theta \in R & , \end{array} \tag{2.7}$$

and the same thing for $V_\alpha^\pm$. In this case invariant functions under the above $U(1)$ symmetry are generated by

$$\prod_i U_{\alpha_i}^+ U_{\beta_i}^- \quad . \tag{2.8}$$

Simple examples are given by the two following copies of S^3 spheres,

$$\begin{aligned} U^{+\alpha} U_\alpha^- &= \left|U^{+1}\right|^2 + \left|U^{+2}\right|^2 = p \quad , \\ U^{\pm\alpha} U_\alpha^\pm &= 0 \quad , \qquad p = r_1^2 \quad , \end{aligned} \tag{2.9}$$

embedded in C_u^2 and

$$\begin{aligned} V_\beta^- V^{+\beta} &= \left|V_1^-\right|^2 + \left|V_2^-\right|^2 = q \quad , \\ V^{\pm\beta} V_\beta^\pm &= 0 \quad , \qquad q = r_2^2 \quad , \end{aligned} \tag{2.10}$$

embedded in C_v^2. The relations $U^{\pm\alpha}U_\alpha^\pm = 0$ and $V^{\pm\beta}V_\beta^\pm = 0$ reflect that $U_\alpha^\pm$ and $V_\alpha^\pm$ are commuting isodoublets.

In this harmonic frame work, the usual conifold algebraic geometry equation $xy - zw = \mu$, with $x, y, z, w \in C$, takes the following $SL(2, C)$ covariant form,

$$U^{+\alpha} V_\alpha^- = \varepsilon^{\alpha\beta} U_\beta^+ V_\alpha^- = \mu \quad , \qquad \varepsilon^{12} = -\varepsilon^{21} = 1 \quad , \tag{2.11}$$

where there is no place to the complex conjugate variables $V^{+\alpha}$ and U_α^- and where now the pair of holomorphic complex variables $U^{+\alpha}$ and V_α^- form a $SL(2, C)$ doublet,

$$\begin{pmatrix} U^+ \\ V^- \end{pmatrix} \quad . \tag{2.12}$$

The generators ∇^{++}, ∇^{--} and $\nabla^0 = \left[\nabla^{++}, \nabla^{--}\right]$ of the $SL(2, C)$ rotations of above harmonic variables are given by

$$\begin{aligned} \nabla^{++} &= U^{+\alpha} \frac{\partial}{\partial V^{-\alpha}} \quad , \\ \nabla^{--} &= V^{-\alpha} \frac{\partial}{\partial U^{+\alpha}} \quad , \\ \nabla^0 &= \left(U^{+\alpha} \frac{\partial}{\partial U^{+\alpha}} - V^{-\alpha} \frac{\partial}{\partial V^{-\alpha}} \right) \quad . \end{aligned} \tag{2.13}$$

We also have the following typical relations,

$$U^+ = \nabla^{++}\left(V^-\right) \quad , \qquad V^- = \nabla^{--}\left(U^+\right) \quad . \tag{2.14}$$

A direct comparison between the usual conifold expression $xy - zw = \mu$ and the harmonic one $U^{+\alpha}V_\alpha^- = \mu$ shows that

$$U^{+\alpha} = (x, z) \quad , \tag{2.15}$$

and

$$V_\alpha^- = (y, w) \quad . \tag{2.16}$$

Moreover by setting $V_\alpha^- = \frac{p}{\mu} U_\alpha^-$, the equation $U^{+\alpha}V_\alpha^- = \mu$ reduces to the 3-sphere $U^{+\alpha}U_\alpha^- = p$ while imposing $U^{+\alpha} = \frac{q}{\mu} V^{+\alpha}$ one falls on the second 3-sphere $V^{+\alpha}V_\alpha^- = q$.

2.1.2 Link to 2D $c = 1$ string

Beside the manifest $SL(2)$ covariance, the harmonic variables $U^{+\alpha}$ and V_α^- get a remarkable interpretation in the $c = 1$ string ground ring analysis of Witten [16]. They coincide exactly with the basic conformal spin zero and ghost number zero vertex operators $O_{\frac{1}{2},\frac{\pm 1}{2}} \times \overline{O}_{\frac{1}{2},\frac{\pm 1}{2}}$ generating the ground ring $\mathcal{V} = \mathcal{R} \otimes \overline{\mathcal{R}}$ of the 2D $c = 1$ string theory with non zero cosmological term. More precisely, we have the result,

$$\begin{aligned} U^{+\alpha} &\sim O_{\frac{1}{2},\frac{+1}{2}} \times \overline{O}_{\frac{1}{2},\frac{\pm 1}{2}} \quad , \\ V_\alpha^- &\sim O_{\frac{1}{2},\frac{-1}{2}} \times \overline{O}_{\frac{1}{2},\frac{\mp 1}{2}} \quad , \end{aligned} \tag{2.17}$$

where $\alpha = 1, 2$ $(\alpha = \frac{3}{2} \mp \frac{1}{2})$. Using this correspondence and the $c = 1$ string interpretation from [16, 15], one learns that the harmonic variables $U^{+\alpha}$ are associated with positive left moving units of momenta at the $SU(2)$ radius and V_α^- with negative left moving ones. From the complex three dimension conifold view, the upper components U^{+1} and V_1^- are respectively associated with positive and negative momentum units while the down components U^{+2} and V_2^- are associated with positive and negative winding mode units. This property shows that momentum and winding units of same sign form $SU(2, C)$ doublets while momentum and winding units are rotated under full $SL(2, C)$ symmetry.

2.2 Conifold in harmonic framework

One of the power of $SL(2, C)$ harmonic analysis we are developing here is that isometries of the 3-sphere and the deformed conifold get simple realizations. For the conifold $U^{+\alpha} V_\alpha^- = \mu$, the transformations generating isometries are given by the product of two $SU(2)$ symmetries, that is a full isometry group given by $SU_u(2) \times SU_v(2) \sim SL(2, C)$.

2.2.1 $SU_u(2)$ isometry subgroup

The first $SU(2)$ isometry factor leaving $U^{+\alpha} V_\alpha^- = \mu$ invariant has a group parameter $\Lambda^{++} = \Lambda^{++}(U^+, V^-)$ with the leading harmonic expansion term,

$$\begin{aligned} \Lambda^{++} &= U_{(\alpha}^+ U_{\beta)}^+ \Lambda^{(\alpha\beta)} + ... \\ U_{(\alpha}^+ U_{\beta)}^+ &= \frac{1}{2}\left(U_\alpha^+ U_\beta^+ + U_\beta^+ U_\alpha^+\right) \quad , \end{aligned} \tag{2.18}$$

and acts on the harmonic U^+ and V^- variables as

$$U^{+\alpha} \rightarrow U^{+\alpha\prime} = V^{-\alpha} \Lambda^{++} \quad , \qquad V_\alpha^- \rightarrow V_\alpha^{-\prime} = V_\alpha^- \quad . \tag{2.19}$$

As we see, this transformation leaves V_α^- invariant; and because of the property $V^{-\alpha}V_\alpha^- = 0$, it also leaves invariant the conifold relation $U^{+\alpha}V_\alpha^- = \mu$. We also have the relations

$$\Lambda^{++} = U^{+\alpha}U_\alpha^{+\prime} \quad , \qquad U^{+\alpha\prime}U_\alpha^{+\prime} = 0 \quad , \qquad V_\alpha^{-\prime}V^{-\alpha} = 0 \quad . \tag{2.20}$$

2.2.2 $SU_v(2)$ isometry

The second $SU(2)$ isometry factor operates in the V^--space leaving the variable U^+ invariant. It has a group parameter Γ^{--} and acts as

$$U^{+\alpha} \rightarrow U^{+\alpha\prime\prime} = U^{+\alpha} \quad , \qquad V_\alpha^- \rightarrow V_\alpha^{-\prime\prime} + U_\alpha^+\Gamma^{--} \quad , \tag{2.21}$$

leaving invariant conifold defining equation since $U^{+\alpha}U_\alpha^+ = 0$. Like in the harmonic superspace formulation of extended supersymmetry [33], one discovers here also that reality of the group parameters Λ^{++} and Γ^{--} should be thought of as

$$\Lambda^{++} = \widetilde{\Lambda^{++}} \quad , \qquad \Gamma^{--} = \widetilde{\Gamma^{--}} \quad , \tag{2.22}$$

where $(\sim)$ stands for a combination of the usual complex conjugation $(-)$ and the conjugation $(*)$ of the charge of the $U_C(1)$ Cartan Weyl subgroup of $SU(2)$.

2.3 Partition function $\mathcal{Z}_{top}\left(T^*S^3\right)$

In the usual explicit expression of the partition function of the B model topological string on conifold, the function $\mathcal{Z}_{top}(T^*S^3)$, with its standard factorisation,

$$\mathcal{Z}_{top}(t) = \exp\left(\sum_{g=0}^{\infty}\left(\frac{\mu}{g_s}\right)^{2-2g}\mathcal{F}_g(t)\right) \quad , \tag{2.23}$$

is given by sums involving monomials $\prod_{i=1}^{k} t_{n_i}\prod_{j=1}^{l}\widetilde{t}_{-m_j}$ in the modes t_n and $\widetilde{t}_{-n}$, $(n = 2s \in \mathbb{N}^*)$ of the local complex deformation moduli $t = t(x,y,z,w)$ of the deformed conifold

$$xy - zw = \mu + t(x,y,z,w) \quad . \tag{2.24}$$

In case where T^*S^3 is restricted to S^3, the modes $\widetilde{t}_{-n}$ coincides with $\overline{t_n} = t_{-n}$; the complex conjugates of t_n. These t_n modes are generally taken as,

$$t_{\pm n} = \int_{S^1} e^{-in\theta}t(x,\overline{x}) \quad , \qquad n = 2s = 1,2,... \quad , \tag{2.25}$$

where we have solved eq $xy - zw = \mathrm{Re}\,\mu$ by setting

$$z = w = 0 \tag{2.26}$$

and taking

$$
\begin{aligned}
x &= \sqrt{\operatorname{Re}\mu}\exp i\theta \quad , \\
y &= \sqrt{\operatorname{Re}\mu}\exp\left(-i\theta\right) \quad .
\end{aligned}
\tag{2.27}
$$

From the $c=1$ string ground ring view of [16, 30], these moduli are associated with the conformal vertex operators

$$
W^{+}_{s,\pm s} = e^{\left[\pm is\sqrt{2}X+(1+s)\sqrt{2}\varphi\right]} \quad , \tag{2.28}
$$

involving tachyons $V_{s,\pm s}\left(X\right)\sim\exp\left(\pm is\sqrt{2}X\right)$ with discrete momentum p_s fixed as $p_s=\pm s\sqrt{2}$ $(s=\frac{n}{2})$ together with the vertex operators $\exp\left(\left(1+s\right)\sqrt{2}\varphi\right)$ based on Liouville field φ. The ground ring $\mathcal{V}=\mathcal{R}\otimes\overline{\mathcal{R}}$ of the $c=1$ string theory deals with discrete primary fields effectively described by the vertex operators $V_{s,\pm n}\left(X\right)$ with discrete momenta p_n given by,

$$
p_n=\pm n\sqrt{2} \quad , \qquad |n|\leq s-1 \quad . \tag{2.29}
$$

As we will see later, these operators which, from conifold view, may be interpreted as describing couplings between momenta and winding modes, have a natural description in the harmonic analysis. The previous Fourrier modes t_{+n} and t_{-n} get respectively replaced by the homogeneous harmonic functions

$$
\zeta^{+n}=U^{+}_{(\alpha_1}...U^{+}_{\alpha_n)}\zeta^{(\alpha_1...\alpha_n)} \quad , \tag{2.30}
$$

and

$$
\zeta^{-n}=U^{-}_{(\alpha_1}...U^{-}_{\alpha_n)}\overline{\zeta}^{(\alpha_1...\alpha_n)} \quad , \tag{2.31}
$$

living on the 2-sphere $S^2\sim S^3/S^1$. The point is that infinitesimally, the 3-sphere described by $U^{+\alpha}U^{-}_{\alpha}=p$ is deformed as

$$
U^{+\alpha}U^{-}_{\alpha}=p+\xi\left(U^{+},U^{-}\right) \quad . \tag{2.32}
$$

As a real function, the local complex deformation function ξ reads as

$$
\xi\left(U^{+},U^{-}\right)\sim\zeta\left(U^{+}\right)+\overline{\zeta}\left(U^{-}\right)+O\left(\left[\zeta+\overline{\zeta}\right]^2\right) \tag{2.33}
$$

and may be Fourrier expanded, at first order in ζ and $\overline{\zeta}$, as

$$
\xi\left(U^{+},U^{-}\right)\sim\sum_{n\geq 0}\left(e^{in\theta}\zeta^{+n}+e^{-in\theta}\overline{\zeta}^{-n}\right)+\mathcal{O}\left(\left[\zeta+\overline{\zeta}\right]^2\right) \quad , \tag{2.34}
$$

where ζ^{+n} and $\overline{\zeta}^{-n}$ are as in eqs(2.30-2.31). Note in passing that eq(2.30) reads more explicitly as,

$$
\begin{aligned}
\zeta^{+n}\left(U^{+}\right) &= \left(U^{+}_1\right)^n\ \zeta^{(1...1)}+\left(U^{+}_1\right)^{n-1}\left(U^{+}_2\right)\ \zeta^{(1...12)}+... \\
&\quad +\left(U^{+}_1\right)\left(U^{+}_2\right)^{n-1}\ \zeta^{(12...2)}+\left(U^{+}_2\right)^n\ \zeta^{(2...2)} \quad .
\end{aligned}
\tag{2.35}
$$

By using the identification $x = U^{+1}$ and $z = U^{+2}$, one sees that there is a $1 \rightarrow 1$ correspondence between the $t_{\pm n}$ Fourrier modes and $\zeta^{\pm n}$ harmonic functions on S^2; and a $1 \rightarrow (n+1)$ correspondence between t_{+n} and the multiplets $\zeta^{(\alpha_1 \ldots \alpha_n)}$. The particular mode $\zeta^{(2\ldots 2)}$ corresponds, up to a sign, to t_n; while the remaining (n) modes $\zeta^{(\alpha_1 \ldots \alpha_n)}$ with all $\alpha_i \neq 2$ have however no analogue in the usual formulation. The reason is that in the standard approach, winding modes are ignored.

The correspondence between Fourrier analysis on S^1 and harmonic one on S^3 is in fact more general. For instance the integral on periodic functions living on S^1,

$$\int_{S^1} [t(\theta)]^2 \sim 2 \sum_{n \geq 1} t_n \overline{t}_{-n} \quad , \tag{2.36}$$

has a counterpart in harmonic analysis on the 3-sphere. We have

$$\int_{S^3} \left[\xi\left(u^+, u^-\right)\right]^2 \sim 2 \sum_{n \geq 1} \left(\int_{S^2} \zeta^{+n} \overline{\zeta}^{-n} \right) \quad , \tag{2.37}$$

which, by integration on the harmonic variables of the 2-sphere using harmonic integration rules given in appendix, we obtain,

$$\int_{S^3} \left[\xi\left(u^+, u^-\right)\right]^2 \sim 2 \sum_{n \geq 1} \frac{1}{(n+1)} \zeta^{(\alpha_1 \ldots \alpha_n)} \overline{\zeta}_{(\alpha_1 \ldots \alpha_n)} \quad , \tag{2.38}$$

where now $\zeta^{(\alpha_1 \ldots \alpha_n)}$ are $SU(2)$ tensors with spin $s = (n+1)$ and $\overline{\zeta}_{(\alpha_1 \ldots \alpha_n)}$ their complex conjugates. Note that the coefficient $\frac{1}{(n+1)}$ in RHS of above eq shows that in a $SL(2)$ manifestly covariant formulation, each component of the trace $\zeta^{(\alpha_1 \ldots \alpha_n)} \overline{\zeta}_{(\alpha_1 \ldots \alpha_n)}$ contributes with the same weight. As such, the monomials type $\prod_{i=1}^{k} t_{n_i} \prod_{j=1}^{l} \widetilde{t}_{-m_j}$, with $n_1 + \ldots + n_k = m_1 + \ldots + m_l$ which are involved in the explicit expression of free energy $\mathcal{F} = \mathcal{F}\left(t, \widetilde{t}\right)$ of topological string B model restricted to the 3-sphere, should be thought of as given by

$$\int_{S^3} \left[\zeta\left(U^+\right) + \overline{\zeta}\left(U^-\right)\right]^k = \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!} \int_{S^3} \left[\zeta\left(U^+\right)\right]^{k-j} \left[\overline{\zeta}\left(U^-\right)\right]^j \quad , \tag{2.39}$$

as required by the manifestly $SU(2)$ invariant harmonic analysis. In this manner, one can immediately determine the $SL(2)$ covariant expression of the partition function

$$\mathcal{Z}_{top}\left(\zeta, \widetilde{\zeta}\right) = \exp\left(\sum_{g=0}^{\infty} \left(\frac{\mu}{g_s}\right)^{2-2g} \mathcal{F}_g\left(\zeta, \widetilde{\zeta}\right) \right) \quad . \tag{2.40}$$

The rule is as follows: (**i**) start from the usual expression of $\mathcal{Z}_{top} = \mathcal{Z}_{top}\left(t, \widetilde{t}\right)$, (**ii**) make the substitution

$$\left(t, \widetilde{t}\right) \qquad \longrightarrow \qquad \left(\zeta, \widetilde{\zeta}\right) \quad , \tag{2.41}$$

and (**iii**) perform traces on $SL(2)$ representations. The result we get is as in eqs(1.3); for details see section 6.

With this rule, one can extend all results obtained for the large circle S^1 of the 3-sphere to the whole points of S^3; see also theorem of subsubsection 6.2.2.

Using the correspondence between Fourrier analysis on S^1 and the harmonic one on S^3, we reconsider in section 7 the cosmological toy model of GSV; in particular the computation of the manifestly $SU(2)$ covariant N points Green functions

$$G_N = < \Phi_1 ... \Phi_N > \quad , \tag{2.42}$$

of the conformal factor Φ_i describing the correlations of the fluctuations of the S^3 real slice of the conifold T^*S^3. In particular, we show that quantum correlations are generated by the functional,

$$\mathcal{Z}\left[J, \overline{J}\right] = \mathcal{N} \int_{\mathcal{M}} D\zeta D\overline{\zeta} \ \exp\left(-\frac{1}{g_s^2}\mathcal{S}\left[p, \zeta, \overline{\zeta}\right] + \int_{S^3}\left(J\xi + \overline{J\zeta}\right)\right) \quad , \tag{2.43}$$

where g_s is the usual string coupling, $\sqrt{p}$ is the radius of the 3-sphere S^3 described by $U^{+\alpha}U_\alpha^- = p$ and $\mathcal{M}$ the moduli space of local complex deformations of conifold. In this relation the field variable

$$\xi = \xi\left(S^3\right) \simeq \zeta + \overline{\zeta} \quad , \tag{2.44}$$

describes the infinitesimal complex deformations of the 3-sphere and $\mathcal{S}\left[p, \zeta, \overline{\zeta}\right]$ is given by the following functional.

$$\mathcal{S}\left[p, \zeta, \overline{\zeta}\right] = \int_{S^3} \overline{\zeta} \frac{2}{D^{++}D^{--} + D^{--}D^{++}} \zeta - \frac{1}{3p}\int_{S^3}\left(\zeta + \overline{\zeta}\right)^3 + O\left(\frac{1}{p^2}\xi^4\right) \quad , \tag{2.45}$$

where D^{++}, D^{--} and $[D^{++}, D^{--}] = D^0$ generate the $SU(2, C)$ algebra and where the term $O\left(\frac{1}{p^2}\xi^4\right)$ stands for higher fluctuations field variable interactions. Besides the fact that ξ describes small fluctuations around p, the above mentioned approximation may be also justified in the S^3 cosmology model where the volume of S^3 is supposed large $(p \to \infty)$.

In the end of this presentation, we would like to add that in harmonic frame work, special features on topological theory becomes manifest. For instance, the explicit red-erivation of topological Chern Simons gauge theory from the gauging of conifold C^* symmetry [32] is one of these features. An other example concerns conifold local complex deformation parameter ξ. A careful analysis shows that the right way to express the local conifold equation is as $\xi = \nabla^{++}\xi^{--} + \nabla^{--}\xi^{++}$ where now there is no zero mode μ, a property which is not exactly fulfilled in above discussion. If we substitute above ξ as $\nabla^{++}\xi^{--} + \nabla^{--}\xi^{++}$, one ends with the

$$\int_{S^3} \xi^{--}\xi^{++} \quad , \tag{2.46}$$

instead of $\int_{S^3} \overline{\zeta} \frac{2}{\{D^{++},D^{--}\}} \zeta$ where apparent non localities disappear. We leave these details and others to the developments of forthcoming sections.

3 c=1 string theory and Conifold geometry

In the first part of this section, we review briefly 2D target space $c = 1$ string theory; in particular its quantum ground ring of the discrete primary states at the $SU(2)$ radius and its connection with conifold geometry. In the second part, we study particular aspects of the conifold T^*S^3 with special focus on its local complex deformations by using standard complex analysis. This description is useful to fix ideas and to make contact with the harmonic frame work introduced in section 2 and which we shall develop further in section 4.

3.1 Discrete primaries in 2D $c = 1$ string theory

As noted before, string theory with two dimensional target space time is characterized by the existence of a special set of discrete primary fields generating three types of ground rings denoted here below as $\mathcal{R}$ for chiral ring, $\overline{\mathcal{R}}$ for antichiral one and $\mathcal{V} = \mathcal{R} \otimes \overline{\mathcal{R}}$ for their combination.

3.1.1 2D $c = 1$ string

To get an idea on the structure of these ground rings, it is interesting to start by recalling that the world sheet action $S[X, \varphi, h] = S$ describing 2D $c = 1$ string theory is,

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left(h^{ij} \partial_i X \partial_j X + h^{ij} \partial_i \varphi \partial_j \varphi - 2\sqrt{\alpha'} \varphi R^{(2)} \right) \\ &\quad + \mu \int d^2\sigma \sqrt{h} \exp\left(-\frac{2}{\sqrt{\alpha'}} \varphi \right) \quad , \end{aligned} \tag{3.1}$$

where $X(\sigma) = X(\sigma^0, \sigma^1)$ is the matter scalar field, $h(\sigma)$ is the world sheet metric, $R^{(2)}(\sigma)$ the Ricci scalar, $\varphi(\sigma)$ the usual Liouville field and μ the cosmological constant. The total holomorphic energy momentum tensor $T = T(X, \varphi, c, b)$ of the quantum theory including $\{b, c\}$ ghost system is,

$$T = T(X) + T(\varphi) + T_{ghost}(c, b) \quad , \tag{3.2}$$

with

$$\begin{aligned} T(X) &= -\frac{1}{2} (\partial X)^2 \quad , \\ T(\varphi) &= -\frac{1}{2} (\partial \varphi)^2 + \varphi \sqrt{2} \quad , \\ T_{ghost}(c, b) &= -2b\partial c - (\partial b) c \quad . \end{aligned} \tag{3.3}$$

One of the interesting things regarding this 2D quantum field theory is that along with the standard vertex operators

$$V_p\left(X\right) = e^{ipX} \quad , \qquad p \in R \quad , \tag{3.4}$$

of conformal weight $h = p^2/2$, there are additional discrete primary fields

$$V_{n/\sqrt{2}}\left(X\right) , \qquad n \text{ an integer} , \tag{3.5}$$

that are involved in the building of the above mentioned ground rings. These $V_{n/\sqrt{2}}$ conformal states belong to representations of the holomorphic (resp. antiholomorphic) $SU\left(2\right)$ current algebra generated by the field operators

$$T^{\pm} = e^{\pm iX\sqrt{2}} \quad , \qquad T^0 = i\sqrt{2}\partial X \quad , \tag{3.6}$$

(resp. $\overline{T^{\pm}}$ and $\overline{T^0}$). As required by the $SU\left(2\right)$ compactification condition $X \rightarrow X + 2\pi$, the momenta p of the conformal vertex $V_p\left(X\right)$ is quantized as

$$p = \frac{n}{\sqrt{2}} \quad , \tag{3.7}$$

with n an integer. Viewed from the $SU\left(2\right)$ holomorphic current algebra, the $V_{n/\sqrt{2}}\left(X\right)$ field operators are generally ranged into $SU\left(2\right)$ spin s representation with highest weight state

$$V_{s,s}\left(X\right) = e^{is\sqrt{2}X} \quad . \tag{3.8}$$

This field operator satisfy the following OPE relations,

$$\begin{aligned} T^+\left(z_1\right) V_{s,s}\left(X\left(z_2\right)\right) &\sim 0 + \text{ regular terms} \quad , \\ T^0\left(z_1\right) V_{s,s}\left(X\left(z_2\right)\right) &\sim \frac{2s}{z_1 - z_2} V_{s,s}\left(X\left(z_2\right)\right) + \text{ regular terms} \quad , \end{aligned} \tag{3.9}$$

where we have set $|n| = 2s$, that is $s = 0, \frac{1}{2}, 1, ...$ and, for simplicity, we re-named the vertex operators $V_{\pm s\sqrt{2}}\left(X\right)$ as $V_{s,\pm s}$ with first index s referring to the $SU\left(2\right)$ spin and the second one, namely $\pm s$, to momentum. In this view, momenta are also the eigenvalues of the charge operator T^0. By repeatedly acting with lowering operators, we get a set of $2s + 1$ discrete conformal fields

$$\left\{V_{s,n}\left(X\right) , \ 0 \leq |n| \leq s\right\} \quad , \tag{3.10}$$

forming an $su\left(2\right)$ spin s multiplet having a conformal dimension $h = s^2$. The special vertices with $|n| = s$ namely

$$V_{s,\pm s}\left(X\right) = e^{\pm is\sqrt{2}X} \quad , \tag{3.11}$$

are standard tachyon operators with momenta $p = \pm s\sqrt{2}$ while the remaining others $V_{s,n}$ are new objects having momenta $|n| < s$. For the particular cases $s = 0$ and $s = \frac{1}{2}$, the corresponding vertex operators are associated with the identity operator

$$V_{0,0} = I \quad , \tag{3.12}$$

and the two component spinor vertex $V_{\frac{1}{2},\pm\frac{1}{2}}$ given by,

$$V_{\frac{1}{2},+\frac{1}{2}} = e^{+\frac{i}{\sqrt{2}}X} \quad , \qquad V_{\frac{1}{2},-\frac{1}{2}} = e^{-\frac{i}{\sqrt{2}}X} \quad . \tag{3.13}$$

They have conformal weights $h_{1/2} = \frac{1}{4}$.

3.1.2 Quantum field operators

The new $V_{s,n}$ states when coupled to the operators $V_\omega(\varphi) = e^{i\omega\varphi}$ with conformal dimension $h_\varphi = \left(\omega^2/2 + i\omega\sqrt{2}\right)$ allow to build spin one primary fields $W^+_{s,n} = W^+_{s,n}(X,\varphi)$. Using the vertex operators $e^{(1\pm s)\varphi}$ of conformal dimension

$$h_\varphi = 2(1 \pm s) - (1 \pm s)^2 = 1 - s^2 \quad , \tag{3.14}$$

it is clear that the following holomorphic operators,

$$\begin{aligned} W^+_{s,n} &= V_{s,n} \times e^{(1+s)\sqrt{2}\varphi} \quad , \qquad \omega_+ = \frac{1+s}{i} \quad , \\ W^-_{s,n} &= V_{s,n} \times e^{(1-s)\sqrt{2}\varphi} \quad , \qquad \omega_- = \frac{1-s}{i} \quad , \end{aligned} \tag{3.15}$$

have conformal weights $\left(h, \overline{h}\right) = (1, 0)$ and momentum $\left(n\sqrt{2}, \frac{1+s}{i}\sqrt{2}\right)$ and $\left(n\sqrt{2}, \frac{1-s}{i}\sqrt{2}\right)$ respectively. Note in passing that

$$W^\pm_{0,0} = e^{\sqrt{2}\varphi} \quad , \tag{3.16}$$

it is just the cosmological vertex operator involved in the action (3.1) and,

$$W^-_{\frac{1}{2},\frac{1}{2}} = \exp\left(\frac{i(X + i\varphi)}{\sqrt{2}}\right) \quad , \qquad W^-_{\frac{1}{2},\frac{-1}{2}} = \exp\left(\frac{-i(X - i\varphi)}{\sqrt{2}}\right) \quad , \tag{3.17}$$

are basic operators whose role will be exhibited later on. True quantum field operators are obtained by paring holomorphic sector with the antiholomorphic one. Such a pairing may be achieved however in two manners. First by using standard combination

$$Z^\pm_{s,n,m} = W^\pm_{s,n} \times \overline{W}^\pm_{s,m} \quad , \tag{3.18}$$

of the $\left(h, \overline{h}\right) = (1, 0)$ holomorphic operators $W^\pm_{s,n}$ with the corresponding $\left(h, \overline{h}\right) = (0, 1)$ antiholomorphic ones $\overline{W}^\pm_{s,n}$. These vertex operators have conformal weights $\left(h, \overline{h}\right) = (1, 1)$ and are used in the study of infinitesimal deformations of above CFT_2.

The second class concerns making true quantum field operators $J_{s,n,m}$ and $\overline{J}_{s,n,m}$ by combining holomorphic and antiholomorphic quantities, but still with conformal weights $\left(h,\overline{h}\right) = (1,0)$ and $\left(h,\overline{h}\right) = (0,1)$ respectively. More precisely, for the case of the specific $(1,0)$ operators $J_{s,n,m}$ (resp $(0,1)$ operators $\overline{J}_{s,n,m}$) generating a Lie algebra of symmetries, we have to combine $W^{+}_{s,n}$ (resp $\overline{W}^{+}_{s,n}$) with conformal spin zero fields $\overline{O}_{s-1,n}$ (resp.$O_{s-1,n}$) as shown below,

$$\begin{aligned} J_{s,n,m} &= W^{+}_{s,n} \times \overline{O}_{s-1,m} \quad , \\ \overline{J}_{s,n,m} &= O_{s-1,n} \times \overline{W}^{+}_{s,m} \quad . \end{aligned} \tag{3.19}$$

In these relations, the operators $O_{s-1,n}$ and their antiholomorphic counterpart are the BRST partners of

$$Y^{+}_{s,n} = \mathrm{c}\times W^{+}_{s,n} \quad , \qquad |n| < s \quad , \tag{3.20}$$

which are conformal operators with spin zero and ghost number $G = 1$. Indeed starting from the vertex operators $W^{\pm}_{s,n}$ (resp $\overline{W}^{\pm}_{s,n}$) and using the ghost fields $\mathrm{b} = \mathrm{b}_{zz}$ and $\mathrm{c} = \mathrm{c}^{z}$ of spins 2 and -1, we can build the conformal spin zero and ghost number $G = 1$ operators $Y^{\pm}_{s,n}$ as follows

$$Y^{\pm}_{s,n} = \mathrm{c}\times W^{\pm}_{s,n} \quad , \qquad |n| < s \quad . \tag{3.21}$$

Cohomology of BRST operator Q shows that the $Y^{\pm}_{s,n}$ operators should have partners $QY^{\pm}_{s,n}$ with ghost numbers $G' = G \pm 1$. Following [16], $Y^{+}_{s,n}$ field operators have partners $O_{u,n}$ with $u = s-1$ and momenta $|n| \leq u$ at ghost number $G = 0$ and $Y^{-}_{s,n}$ have partners at $G = 2$. For the leading value $s = 1$ $(u = 0)$, the partner $O_{0,0}$ of $Y^{+}_{1,0}$ is just the identity operator I. For the next value $s = \frac{3}{2}$, we have two discrete primaries $Y^{+}_{\frac{3}{2},\frac{1}{2}}$ and $Y^{+}_{\frac{3}{2},-\frac{1}{2}}$ of spin zero and ghost number $G = 1$. The corresponding spin zero and ghost number $G = 0$ operators $O_{\frac{1}{2},\frac{1}{2}}$ and $O_{\frac{1}{2},\frac{-1}{2}}$ are given by,

$$\begin{aligned} O_{\frac{1}{2},\frac{1}{2}} &= \left(\mathrm{cb} + \frac{i}{\sqrt{2}}\left(\partial X - i\partial\varphi\right)\right) W^{-}_{\frac{1}{2},\frac{1}{2}} \quad , \\ O_{\frac{1}{2},\frac{-1}{2}} &= \left(\mathrm{cb} - \frac{i}{\sqrt{2}}\left(\partial X + i\partial\varphi\right)\right) W^{-}_{\frac{1}{2},\frac{-1}{2}} \quad , \end{aligned} \tag{3.22}$$

where $W^{-}_{\frac{1}{2},\frac{1}{2}}$ and $W^{-}_{\frac{1}{2},\frac{-1}{2}}$ are as in eq(3.17). The left moving momenta $\left(n\sqrt{2}, iu\sqrt{2}\right)$ of these spin zero BRST invariant operators $O_{\frac{1}{2},\frac{\pm 1}{2}}$ are same as for the operators $W^{-}_{\frac{1}{2},\frac{\pm 1}{2}}$ namely $\left(\frac{\pm\sqrt{2}}{2}, \frac{i\sqrt{2}}{2}\right)$. Note that similar relations may be also written down for the operators $\overline{O}_{\frac{1}{2},\frac{\pm 1}{2}}$ involving antiholomorphic sector objects. Note moreover that the spin zero BRST invariant operators $O_{u,n}$ and $\overline{O}_{u,n}$ generate commutative and associative ground rings. The chiral (resp antichiral) ground ring $\mathcal{R}$ $(\overline{\mathcal{R}})$ is generated by the operators $O \equiv O_{u,n}$ (resp $\overline{O} = \overline{O}_{u,n}$) and has a multiplication law given by the short distance product,

$$O\left(z_1\right) O'\left(z_2\right) \sim O''\left(z_2\right) + \{Q, F\} \quad , \tag{3.23}$$

where, like for left hand side, the right hand side of this relation is also BRST invariant. Along with these $\mathcal{R}$ and $\overline{\mathcal{R}}$ ground rings, we have as well the ground ring $\mathcal{V}$ combining left and right movers. It is generated by the field operators,

$$V_{u,n,m} = O_{u,n} \times \overline{O}_{u,m} \quad , \tag{3.24}$$

with a similar multiplication law given by multiplying the left and right moving parts separately. Because of the specific properties of the $O_{\frac{1}{2},\frac{\pm 1}{2}}$ and $\overline{O}_{\frac{1}{2},\frac{\pm 1}{2}}$, the basic generators of the above quantum ring $\mathcal{V}$ are given by the following fundamental objects,

$$\begin{aligned} x &= O_{\frac{1}{2},\frac{1}{2}} \times \overline{O}_{\frac{1}{2},\frac{1}{2}} \quad , \\ y &= O_{\frac{1}{2},\frac{-1}{2}} \times \overline{O}_{\frac{1}{2},-\frac{1}{2}} \quad , \\ z &= O_{\frac{1}{2},\frac{1}{2}} \times \overline{O}_{\frac{1}{2},-\frac{1}{2}} \quad , \\ w &= O_{\frac{1}{2},-\frac{1}{2}} \times \overline{O}_{\frac{1}{2},\frac{1}{2}} \quad , \end{aligned} \tag{3.25}$$

obeying the obvious relation

$$xy - zw = 0 \quad , \tag{3.26}$$

with $x, y, z, w \in C$. As these variable operators turn out to play an important role in the present study, it is interesting to have in mind the following group theoretic property,

operators	x	y	z	w
spin $(s, \overline{s})$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$
Cartan Charge (n, m)	$(1, 1)$	$(-1, -1)$	$(1, -1)$	$(-1, 1)$

, (3.27)

where $(s, \overline{s})$ stand for the spin representation of the $SU(2) \times SU(2)$ symmetry and where (n, m) are the values of the (integer) charges of the Cartan-Weyl operators T^0 and $\overline{T^0}$. Forgetting about the fact that the pairs of complex variables (x, z) and (y, w) are related by complex conjugation and considering complex deformations of the $c = 1$ string theory by the conformal spin $(1, 1)$ operators,

$$Z^+ = \sum_{s,n,m} W^+_{s,n} \times \overline{W}^+_{s,m} \quad , \tag{3.28}$$

the previous three dimensional quadric $xy - zw = 0$ get corrected as follows

$$xy - zw = f(x, y, z, w) \quad , \tag{3.29}$$

where now $f = f(x, y, z, w)$ is a priori a general polynomial transforming under $SU(2) \times SU(2)$ as

$$\bigoplus_{2s,2\overline{s}=0}^{\infty} (s, \overline{s}) \quad . \tag{3.30}$$

For the particular case where the deformation parameter f is a constant, one has an interesting field theoretic interpretation. The constant f corresponds just to the cosmological constant μ $(f = \mu)$ transforming as a $SU(2) \times SU(2)$ singlet. Therefore at $\mu \neq 0$, the quantum ground ring of the 2D $c = 1$ string theory at the $SU(2)$ point is given by the ring of polynomials on conifold T^*S^3

$$xy - zw = \mu \quad . \tag{3.31}$$

The generic case where f is an arbitrary local function is discussed below.

3.2 Geometry of the ground ring

From the point of view of the algebraic geometry of the three dimension conifold, the parameter μ is a complex deformation of the conic singularity $xy - zw = 0$. It is a globally defined quantity since it is independent of the local coordinates of T^*S^3; i.e,

$$\frac{\partial \mu}{\partial x^i} = 0, \qquad x^i = x, y, z, w \quad . \tag{3.32}$$

To exhibit more clearly the isometries of the conifold geometry, it is interesting to think about the ambient complex four space C^4 where lives T^*S^3 as the set of complex 2×2 matrices $\mathcal{M}(2, C)$. In this representation, the complex holomorphic vector (x, y, z, w) is now parameterized as

$$X = \begin{pmatrix} x & w \\ z & y \end{pmatrix} \quad , \tag{3.33}$$

and the conifold defining eq(3.31) takes then the remarkable form

$$\det X = \mu \quad . \tag{3.34}$$

Clearly, this relation has a manifest $GL(2, C) \sim GL(1, C) \times SL(2, C)$ automorphism symmetry acting, through arbitrary matrices M of $GL(2, C)$, as follows,

$$X \to M X M^{-1} \quad , \qquad M \in GL(2, C) \quad , \tag{3.35}$$

Note that along with above eqs (3.31, 3.34), there are also other remarkable representations of the conifold. One of the realizations of this quadric is that obtained by the change of holomorphic variables (x, y, z, w) as $x = (x_1 + ix_2)$, $y = (x_1 - ix_2)$, $z = i(x_3 + ix_4)$ and $w = i(x_3 - ix_4)$, $x_i \in C$. With this holomorphic change the algebraic geometry equation (3.31) reads then as

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sum_{i=1}^{4} x_i^2 = \mu \quad . \tag{3.36}$$

Instead of $SL(2,C)$ invariance[1] of eq(3.34), this equation has rather a manifest $O(4,C)$ symmetry group rotating the holomorphic variables as $x_i' = O_{ij}x_i$ with $O_{ij} \in C$ and $O_{ij}O_{jk} = \delta_{ik}$. Recall also that like in the case of real rotations, here also the complex holomorphic $O(4,C)$ group splits as the product of the holomorphic group product $O(3,C) \times O(3,C)$. The usual trigonometric functions sine and cosine of the real group $O(3,R)$ get now replaced by in the $O(3,C)$ as shown below,

$$\sin\vartheta_j \rightarrow \frac{1}{2i}\left(\lambda_j - \frac{1}{\lambda_j}\right) \quad , \qquad \cos\vartheta_j \rightarrow \frac{1}{2}\left(\lambda_j + \frac{1}{\lambda_j}\right) \quad , \tag{3.37}$$

where the λ_j's are now non zero complex number. To fix the ideas, we give here below the example of the $O(2,C)$ symmetry rotating the x_1 and x_2 variables of the following complex holomorphic relation,

$$x_1^2 + x_2^2 = \nu \quad , \tag{3.38}$$

where ν is some non zero complex number. In this case the 2×2 matrices O leaving (3.38) invariant reads as

$$O = \frac{1}{2}\begin{pmatrix} \lambda + \frac{1}{\lambda} & \frac{i}{\lambda} - i\lambda \\ i\lambda - \frac{i}{\lambda} & \lambda + \frac{1}{\lambda} \end{pmatrix} \quad , \tag{3.39}$$

with general features as,

$$\det O = \pm 1 \quad , \qquad OO^T = O^T O = I \quad . \tag{3.40}$$

Here the $O(2,C)$ group parameter λ, which may be also split by using Euler representation of complex number as $|\lambda| \exp i\vartheta$, is a non zero complex number. Using the change of holomorphic variables $x_1 = (x+y)/2$, $x_2 = (x-y)/2i$, eq(3.38) reads also as

$$xy = \nu \quad . \tag{3.41}$$

This relation is invariant under $GL(1,C) \sim C^*$ transformation generated by the scaling $x \rightarrow \lambda x$ and $y \rightarrow y/\lambda$ with non zero parameter $\lambda = \varrho_1 + i\varrho_2$. From this description, one clearly see that the holomorphic $SO(2,C)$ group is then just $GL(1,C) \sim C^*$; it should be thought of as the complex generalization of the familiar isomorphism $U(1) \sim SO(2,R)$ dealing with real compact form associated with $|\lambda|^2 = \varrho_1^2 + \varrho_2^2 = 1$. For the non compact case where the group parameters ϱ_1 and ϱ_2 are such that $\varrho_1^2 - \varrho_2^2 = 1$, we have the hyperbolic symmetry $SO(1,1;R)$. This presentation extends naturally to higher dimensional holomorphic orthogonal groups $O(n,C)$ and their representations.

[1] The $GL(1,\mathbb{C})$ abelian subsymmetry of $GL(2,\mathbb{C})$ acts trivially of the complex holomorphic variables x, y, z and w.

Moreover, using the factorisation $SO(4,C) \simeq SL(2,C) \times \widetilde{SL(2,C)}$ allowing to express the complex[2] holomorphic 4-vector x_i as a holomorphic bispinor $x^{\alpha\overline{\beta}}$, the conifold eq(3.31) can be also rewritten as

$$\varepsilon_{\alpha\gamma}\varepsilon_{\overline{\beta}\overline{\delta}}x^{\alpha\overline{\beta}}x^{\gamma\overline{\delta}} = x^{\alpha\overline{\beta}}x_{\alpha\overline{\beta}} = \mu \quad , \qquad \varepsilon_{\alpha\gamma} = -\varepsilon_{\gamma\alpha}, \qquad \varepsilon^{12} = \varepsilon_{21} = 1 \quad , \tag{3.42}$$

where the sum over repeated indices is understood. For later use and also to have more insight into these complex relations, we give here below two realizations of eq(3.31) using first free complex holomorphic coordinates (z_1, z_2, z_3) $(z_i \neq 0)$ and, by help of Euler representation of complex numbers, the corresponding real six dimensional parameterization involving three non compact coordinates (r_1, r_2, r_3) and three compact ones $(\theta_1, \theta_2, \theta_3)$. The free complex holomorphic coordinates representation solving the conifold eq(3.31) reads as,

$$\begin{aligned} x &= \frac{\sqrt{\mu}}{2i} z_1^{\frac{1}{2}} \left(z_3^{\frac{1}{2}} - z_3^{-\frac{1}{2}} \right) \quad , \\ y &= \frac{\sqrt{\mu}}{2i} z_1^{-\frac{1}{2}} \left(z_3^{\frac{1}{2}} - z_3^{-\frac{1}{2}} \right) \quad , \\ z &= -\frac{\sqrt{\mu}}{2} z_2^{\frac{1}{2}} \left(z_3^{\frac{1}{2}} + z_3^{-\frac{1}{2}} \right) \quad , \\ w &= \frac{\sqrt{\mu}}{2} z_2^{-\frac{1}{2}} \left(z_3^{\frac{1}{2}} + z_3^{-\frac{1}{2}} \right) \quad , \end{aligned} \tag{3.43}$$

where the involved square roots $\sqrt{z_i}$ are dictated by boundary conditions of spinors. Under the loop transformation $z_i \to z_i \exp 2i\pi$, $\sqrt{z_i} \to -\sqrt{z_i}$ but the conifold variables x, y, z and w remain invariant. The check of this representation as a free solution of eq(3.31) follows directly from the identity

$$\left(z_3^{\frac{1}{2}} + z_3^{-\frac{1}{2}} \right)^2 - \left(z_3^{\frac{1}{2}} - z_3^{-\frac{1}{2}} \right)^2 = 4 z_3^{\frac{1}{2}} z_3^{-\frac{1}{2}} = 4. \tag{3.44}$$

By setting the three free complex holomorphic variables as $z_i = |z_i| \exp i\theta_i$, $|z_i| \equiv r_i^2$, with θ_i angles chosen as

$$\theta_1 = (\psi - \varphi) \quad , \qquad \theta_2 = (\psi + \varphi) \quad , \qquad \theta_3 = \vartheta \quad , \tag{3.45}$$

where ψ, φ and ϑ are the usual three angle used in the parameterization of the real three

[2] For the real forms of $SO(4,\mathbb{C})$, we have the homomorphisms $SO(4,\mathbb{R}) \sim SU(2,\mathbb{C}) \times SU(2,\mathbb{C})$, $SO(1,3;\mathbb{R}) \sim SL(2,\mathbb{C})$ and $SO(2,2;\mathbb{R}) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

sphere, we get the following realization of the conifold,

$$
\begin{aligned}
x &= \frac{\sqrt{\mu}}{2i} r_1 \left(r_3 e^{\frac{i\vartheta}{2}} - \frac{1}{r_3} e^{\frac{-i\vartheta}{2}} \right) e^{\frac{i}{2}(\psi-\varphi)} \quad , \\
y &= \frac{\sqrt{\mu}}{2ir_1} \left(r_3 e^{\frac{i\vartheta}{2}} - \frac{1}{r_3} e^{\frac{-i\vartheta}{2}} \right) e^{\frac{-i}{2}(\psi-\varphi)} \quad , \\
z &= -\frac{\sqrt{\mu}}{2} r_2 \left(r_3 e^{\frac{i\vartheta}{2}} + \frac{1}{r_3} e^{\frac{-i\vartheta}{2}} \right) e^{\frac{i}{2}(\psi+\varphi)} \quad , \\
w &= \frac{\sqrt{\mu}}{2r_2} \left(r_3 e^{\frac{i\vartheta}{2}} + \frac{1}{r_3} e^{\frac{-i\vartheta}{2}} \right) e^{\frac{-i}{2}(\psi+\varphi)} \quad .
\end{aligned}
\tag{3.46}
$$

For physical interpretation, it is interesting to set,

$$
r_1 = \left(\varrho + \frac{1}{\varrho} \right), \qquad r_2 = \left(\sigma + \frac{1}{\sigma} \right), \tag{3.47}
$$

where together with r_3, the real variables ϱ and σ parameterize the three real non compact T^*S^3 dimensions. The real compact variables ϑ, φ, ψ, which vary in $[0, 4\pi]$, parameterize S^3; that is the real slice of the conifold. Note that besides $SL(2)$ manifest symmetry, the conifold equation (3.31) has moreover discrete symmetries. For instance eq(3.31) remains invariant under the following discrete change,

$$
\begin{aligned}
x &\rightarrow x' = (-)^{k+1} x \quad , \qquad y \rightarrow y' = (-)^{k+1} y \quad , \\
z &\rightarrow z' = (-)^{k} z \quad , \qquad w \rightarrow w' = (-)^{k} w \quad ,
\end{aligned}
\tag{3.48}
$$

with k an arbitrary integer. In the realization (3.46), this corresponds to perform the change

$$
\begin{aligned}
\kappa &\rightarrow \kappa' = \frac{1}{\kappa} \quad , \qquad \varrho \rightarrow \varrho' = \frac{1}{\varrho} \quad , \\
\sigma &\rightarrow \sigma' = \frac{1}{\sigma} \quad , \qquad \vartheta \rightarrow \vartheta' = -\vartheta + 2k\pi \quad ,
\end{aligned}
\tag{3.49}
$$

where one recognizes the usual T duality transformation. According to whether k is odd integer or even integer, the 1-cycles $(\psi - \varphi)$ and $(\psi + \varphi)$ are respectively fixed under the change (3.49). Note also that S^3 is recovered from eqs(3.46) by setting $\kappa = \varrho = \sigma = 1$ and $\chi = 0$.

$$
\begin{aligned}
x &= \sqrt{p} e^{\frac{i}{2}(\psi-\varphi)} \sin\left(\frac{\vartheta}{2}\right) \quad , \\
y &= \sqrt{p} e^{\frac{-i}{2}(\psi-\varphi)} \sin\left(\frac{\vartheta}{2}\right) \quad , \\
z &= -\sqrt{p} e^{\frac{i}{2}(\psi+\varphi)} \cos\left(\frac{\vartheta}{2}\right) \quad , \\
w &= \sqrt{p} e^{\frac{-i}{2}(\psi+\varphi)} \cos\left(\frac{\vartheta}{2}\right) \quad .
\end{aligned}
\tag{3.50}
$$

In fact the condition $\kappa = \varrho = \sigma = 1$ is required by $SU(2)$ unimodularity which demands the identifications $y = \overline{x}$ and $w = -\overline{z}$ so that the matrix X of eq(3.33) takes the form,

$$X = \begin{pmatrix} x & -\overline{z} \\ z & \overline{x} \end{pmatrix} \quad , \tag{3.51}$$

and then eq(3.31) reads as $\det X = |x|^2 + |z|^2 = p$. In group theoretic language, the automorphism symmetry leaving $\det X$ invariant corresponds to the restriction $M^{-1} = M^{+}$ in eq(3.35); that is the group reduction

$$SL(2) \qquad \rightarrow \qquad SU(2) \quad , \tag{3.52}$$

and the abelian complex subgroup C^* contained in $SL(2)$ into the usual Cartan Weyl subgroup $U_C(1)$ of $SU(2)$. Observe also that by setting $\vartheta = \pi$ and $\psi - \varphi = 2\theta$, the 3-sphere reduces to its large large circle $|x|^2 = p$.

3.3 Local deformations and symmetry group representations

In the limit $\mu \rightarrow 0$, the 3-sphere $|x|^2 + |z|^2 = \mathrm{Re}\,\mu$ shrinks to a point and the conifold $xy - zw = 0$ becomes singular. Complex deformations lifting this conic singularity are generally obtained by perturbing the previous relation as

$$xy - zw = F(x, y, z, w) \quad . \tag{3.53}$$

According to the values of F, one distinguishes global deformations captured by the zero mode $\mu = F(x = 0, ..., w = 0)$ and local deformations carried by

$$\mathrm{T}(x, y, z, w) = F(x, y, z, w) - \mu \quad . \tag{3.54}$$

The function $\mathrm{T}(x, y, z, w)$ is a priori an arbitrary holomorphic function on T^*S^3 generated by homogeneous monomials type $x^{n_1}y^{n_2}z^{n_3}w^{n_4}$ as shown below,

$$\mathrm{T}(x, y, z, w) = \sum_{n,m} \sum_{n_1+n_2=n} \sum_{n_3+n_4=m} T_{n_1,-n_2,n_3,-n_4}\ x^{n_1}y^{n_2}z^{n_3}w^{n_4} \quad . \tag{3.55}$$

By grouping monomials with same homogeneous degrees, one can show that the above development may be rearranged in terms of highest weight representations of $SL(2, C)$ conifold isometry. Observe in passing that there are various kinds of $SU(2, C)$ ($SL(2, C)$) subgroups within the homogeneity group $SO(8, R)$ of the real ambient space $R^8 \sim C^4$ where lives the conifold. These group symmetries have a nice description in the harmonic frame work eqs(2.19,2.21). Let us discuss below some of these $SO(8, R)$ subgroups.

3.3.1 $SU(2,C) \times \widetilde{SU(2,C)}$

First of all note that one distinguishes two special $SU(2)$ subgroups of $SO(8,R)$ which deal with the real slice of the conifold. They are generated by the operator sets $\{J_0, J_\pm\}$ and $\left\{\widetilde{J_0}, \widetilde{J_\pm}\right\}$ operating in $x-z$ and $y-w$ complex planes respectively. For the first set, we have the geometric realization,

$$
\begin{aligned}
J_+ &= x\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial \overline{x}} \quad , \qquad (J_+)^\dagger = J_- \quad , \\
J_- &= \overline{z}\frac{\partial}{\partial x} - \overline{x}\frac{\partial}{\partial z} \quad , \qquad (J_0)^\dagger = J_0 \quad , \\
J_0 &= \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right) - \left(\overline{x}\frac{\partial}{\partial \overline{x}} + \overline{z}\frac{\partial}{\partial \overline{z}}\right) \quad ,
\end{aligned}
\tag{3.56}
$$

and for the second the following one,

$$
\begin{aligned}
\widetilde{J_+} &= y\frac{\partial}{\partial \overline{w}} - z\frac{\partial}{\partial \overline{w}} \quad , \qquad \left(\widetilde{J_+}\right)^\dagger = \widetilde{J_-} \quad , \\
\widetilde{J_-} &= \overline{w}\frac{\partial}{\partial y} - \overline{y}\frac{\partial}{\partial w} \quad , \qquad \left(\widetilde{J_0}\right)^\dagger = \widetilde{J_0} \quad , \\
\widetilde{J_0} &= \left(y\frac{\partial}{\partial y} + w\frac{\partial}{\partial w}\right) - \left(\overline{y}\frac{\partial}{\partial \overline{y}} + \overline{w}\frac{\partial}{\partial \overline{w}}\right) \quad .
\end{aligned}
\tag{3.57}
$$

These eqs are interchanged by making the substitution $(x,z) \longleftrightarrow (y,w)$.

3.3.2 $SL(2,C) \times \widetilde{SL(2,C)}$

There are also two others $SL(2,C)$ subgroups of $SO(8,R)$ described by the sets $\{L_0, L_\pm\}$ and $\left\{\widetilde{L_0}, \widetilde{L_\pm}\right\}$; they generate the holomorphic $SL(2,C)$ and $\widetilde{SL(2,C)}$ groups involved in the expansion (3.55). Their geometric realizations read as follows,

$$
L_+ = x\frac{\partial}{\partial z} \quad , \qquad L_- = z\frac{\partial}{\partial x} \quad , \qquad L_0 = \left(x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}\right) \quad , \tag{3.58}
$$

and similar relations for $\left\{\widetilde{L_0}, \widetilde{L_\pm}\right\}$ by making the substitution $(x,z) \rightarrow (y,w)$, that is

$$
\widetilde{L_+} = y\frac{\partial}{\partial w} \quad , \qquad \widetilde{L_-} = w\frac{\partial}{\partial y} \quad , \qquad \widetilde{L_0} = \left(y\frac{\partial}{\partial y} - w\frac{\partial}{\partial w}\right) \quad , \tag{3.59}
$$

Under these groups, the homogeneous degree n polynomials

$$
\begin{aligned}
P_n(x,z) &= \sum_{j=1}^{n} a_j x^{n-j} z^j \quad , \\
\widetilde{P}_n(y,w) &= \sum_{j=1}^{n} a_j y^{n-j} w^j ,
\end{aligned}
\tag{3.60}
$$

with positive integer n transform as spin $s=\frac{n}{2}$ ($\widetilde{s}=\frac{n}{2}$) representations of $SL(2,C)$ ($\widetilde{SL(2,C)}$). From this point of view, the development (3.55) can be viewed as an expansion of local complex deformations on the complete set of the irreducible representations $(\underline{\mathbf{s}},\underline{\widetilde{\mathbf{s}}})$ of $SL(2)\times\widetilde{SL(2)}$ as shown below

$$\begin{aligned}\bigoplus_{2s+1,2\widetilde{s}+1=1}^{\infty}(\underline{\mathbf{s}},\underline{\widetilde{\mathbf{s}}}) = &\ (0,0) \oplus \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right) \\ &\oplus (1,0) \oplus \left(\frac{1}{2},\frac{1}{2}\right) \oplus (0,1) \\ &\oplus \left(\frac{3}{2},0\right) \oplus \ldots \ .\end{aligned} \tag{3.61}$$

The leading singlet $(0,0)$ corresponds just to the global modulus μ and remaining others to local deformations giving the parameters of $Diff\,(T^{*}S^{3})$, the group of holomorphic diffeomorphisms of the conifold.

3.3.3 $SL(2,C)_{diag}$

There is moreover an other remarkable holomorphic $SL(2,C)$ isometry group of the conifold. It is generated by transformations that mix the variables (x,z) and (y,w),

$$\begin{aligned}K_{+} &= x\frac{\partial}{\partial w}-z\frac{\partial}{\partial y} \quad , \qquad K_{-}=w\frac{\partial}{\partial x}-y\frac{\partial}{\partial z} \quad , \\ K_{0} &= \left(x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}\right)-\left(y\frac{\partial}{\partial y}+w\frac{\partial}{\partial w}\right) \quad .\end{aligned} \tag{3.62}$$

It is this symmetry group which mostly concerns us in the present study. Upon identifying $y=\overline{x}$ and $w=\overline{z}$, the conifold reduces to its S^{3} lagrangian submanifold and $SL(2,C)$ isometry down to its $SU(2,C)$ subgroup, eqs(3.56).

3.4 Reduction to complex one dimension

Generally speaking, the hypersurface $xy-zw=\mu$ represents a particular non compact Calabi-Yau threefold. Typical geometries embedded in C^{4} which are used in practice have in general the form

$$G(z,w,x,y)=zw-H(y,x)=0 \quad , \tag{3.63}$$

with $H(y,x)$ some bi-holomorphic function. These geometries have a conical Ricci flat metric and a holomorphic $(3,0)$ form Ω which can be chosen as

$$\Omega\simeq\frac{dz}{z}\wedge dy\wedge dx \quad , \tag{3.64}$$

where we have used $\partial G/\partial w = z$, or equivalently as $\frac{dw}{w} \wedge dy \wedge dx$ by using (w, y, x) as the local coordinates. A general expression of this 3-form keeping touch with the $z \leftrightarrow w$ permutation symmetry is obviously given by a linear combination type,

$$\Omega(a, b) = \left(\frac{adz + bdw}{az + bw}\right) \wedge dy \wedge dx \quad , \tag{3.65}$$

where the pole singularities are located at $(z, w) = (\pm b, \mp a)$ and where a and b are two constants; one of them should be different from zero ($ab \neq 0$). Forgetting for a while about this detail and focus on the way this holomorphic 3-form is handled in topological string B-model on non compact Calabi-Yau threefolds.

3.4.1 Rational curve $zw = H$

In dealing with Ω, one generally fixes the dependence in z and w and consider only perturbations of the function $H(y, x)$. In this manner, the problem reduces essentially to one complex dimension since the Calabi-Yau threefold is viewed as a fibration over the (x, y) plane with fiber given by the rational curve $zw = H(y, x)$. By using Cauchy's theorem in z-plane, we can perform a partial integration of the period,

$$\int_A \Omega = \int_{\gamma_z \times D} \Omega \quad , \tag{3.66}$$

with 3-cycles A given by $\gamma_z \times D$ (or in general $A = (\gamma_z \times D) \cup (\gamma_w \times D')$). We end with the complex structure

$$\int_D dy \wedge dx \quad , \tag{3.67}$$

given by the integral of the holomorphic 2-form $dy \wedge dx$ on the 2-cycle D. The CY3 is then reduced to the holomorphic curve Σ given by

$$H(y, x) = 0 \quad , \tag{3.68}$$

with analytic domain D and boundary $\partial D \subset \Sigma$. By stockes theorem, the integral $\int_D d(ydx)$ can be reduced further to

$$\int_{\partial D} ydx \quad , \tag{3.69}$$

showing that the complex deformation of the holomorphic curve $H(y, x)$ is controlled by the 1-form ydx. In the case of conifold we are interested in here and where

$$H(y, x) = yx - \mu \quad , \tag{3.70}$$

the above 1-form reads then as $\mu \frac{dx}{x}$. Note that the perturbation of $H(y, x)$ by the local deformations $\tau(x) = \sum_{n>0} t_n x^n$, periods of ydx around the 1-cycle γ_x are given by the modes t_n,

$$t_n \sim \int_{\gamma_x} \frac{dx}{x} x^{-n} \tau(x) \quad , \qquad \gamma_x = \partial_x D \quad . \tag{3.71}$$

These complex deformation moduli transform under the projective change $x \rightarrow \lambda x$ and $y \rightarrow \frac{1}{\lambda} y$ as $t_n \rightarrow \lambda^{-n} t_n$. Note that what we have done for the holomorphic variable x may be equally done for the dual variable y. Instead of eq(3.71), one has rather

$$\widetilde{t}_{-n} \sim \int_{\gamma_y} \frac{dy}{y} y^{-n} \widetilde{\tau}(y) \quad . \tag{3.72}$$

To have both of modes t_n and $\widetilde{t}_{-n}$, one should have a local complex deformation type $H(y, x) = \mu + \tau(x) + \widetilde{\tau}(y)$ letting understand that $\tau(x)$ and $\widetilde{\tau}(y)$ are just the two leading perturbation terms of a two variable holomorphic function

$$t(x, y) = \tau(x) + \widetilde{\tau}(y) + O\left[\tau\widetilde{\tau}, \tau^2, \widetilde{\tau}^2\right] \quad . \tag{3.73}$$

where $O\left[\tau\widetilde{\tau}, \tau^2, \widetilde{\tau}^2\right]$ stands for higher perturbation orders.

3.4.2 Beyond eq(3.73)

One may also consider perturbations involving, in addition to x and y, the z and w variables as well. This is particularly interesting for the conifold case where the two holomorphic variable isodoublets $(x, z) \equiv u^\alpha$ and $(y, w) \equiv v_\alpha$ play a symmetric role. In this case the conifold equation reads as

$$u^\alpha v_\alpha = \mu \quad , \tag{3.74}$$

and the holomorphic 3-form invariant under $SL(2)$ isometry group may be written as,

$$\begin{aligned} \Omega &= \frac{1}{2} \frac{\left(\varepsilon_{\alpha\beta}\eta^\alpha du^\beta\right)}{\eta.u} \wedge \left(dv_\gamma \wedge dv_\delta \varepsilon^{\gamma\delta}\right) \\ &= \frac{\eta_1 dx + \eta_2 dz}{\eta_1 x + \eta_2 z} \wedge dy \wedge dw \quad , \end{aligned} \tag{3.75}$$

where $\eta_\alpha = (\eta_1, \eta_2)$ is a constant isodoublet and where the scalar $\eta.u$ stands for $SL(2, C)$ invariant product

$$\eta_\delta u^\delta = \varepsilon_{\delta\gamma} u^\delta \eta^\gamma \quad . \tag{3.76}$$

The point $u_\delta = \eta_\delta$, corresponding to $x = \eta^1 = \eta_2$ and $z = \eta^2 = -\eta_1$, is a pole singularity of Ω. Performing a partial integration of the period $\int_A \Omega$ by using Cauchy's theorem in the plane

$$z' = \eta_1 x + \eta_2 z \quad , \tag{3.77}$$

one gets the restriction $u_\delta = \eta_\delta$, $\eta^\alpha v_\alpha = \mu$ and ends with

$$\int_D \left(dv_\gamma \wedge dv_\delta\right) \varepsilon^{\gamma\delta} \quad , \tag{3.78}$$

describing the complex structure of the holomorphic 2-form $(dv_\gamma \wedge dv_\delta)\,\varepsilon^{\gamma\delta}$ on the 2-cycle D. By stockes theorem, this integral can be also brought to

$$\int_{\partial D} (v_\gamma dv_\delta)\,\varepsilon^{\gamma\delta} \quad , \tag{3.79}$$

showing that part of complex deformations of the conifold is controlled by the 1-form

$$(v_\gamma dv_\delta)\,\varepsilon^{\gamma\delta} \quad , \tag{3.80}$$

with $\eta^\alpha v_\alpha = \mu$. Solving the constraint eq $\eta^\alpha v_\alpha = \mu$ as

$$\eta^\alpha = \frac{\mu}{\theta.v}\theta^\gamma \quad , \tag{3.81}$$

one may rewrite the complex structure as follows,

$$\frac{(\theta_\gamma dv_\delta)\,\varepsilon^{\gamma\delta}}{\theta.v} = \frac{d\,(\theta.v)}{\theta.v} = \frac{(\theta_1 dv_2 - \theta_2 dv_1)}{\theta_1 v_2 - \theta_2 v_1} \quad , \tag{3.82}$$

where $\theta^\alpha = \left(\theta^1, \theta^2\right)$ is a constant isospinor. In the (x, y, z, w) complex holomorphic variable language, the local complex deformations read infinitesimally as

$$T\,(x, y, z, w) \simeq \tau\,(x) + \widetilde{\tau}\,(y) + \upsilon\,(z) + \widetilde{\upsilon}\,(w) \quad , \tag{3.83}$$

with the mode expansions,

$$\begin{aligned} \tau\,(x) &= \sum_{n>0} t_n x^n, \qquad \widetilde{\tau}\,(y) = \sum_{n>0} t_{-n} y^n \quad , \\ \upsilon\,(x) &= \sum_{n>0} s_n z^n, \qquad \widetilde{\upsilon}\,(y) = \sum_{n>0} s_{-n} w^n \quad . \end{aligned} \tag{3.84}$$

Up on thinking about y and w as $y \sim \frac{1}{x}$ and $z \sim \frac{1}{w}$, eqs(3.84) can be put into the following simplest form,

$$\begin{aligned} \tau\,(x) + \widetilde{\tau}\left(\frac{1}{x}\right) &\sim t\,(x) = \sum_{n\neq 0} t_n x^n \quad , \\ \upsilon\,(z) + \widetilde{\upsilon}\left(\frac{1}{z}\right) &\sim s\,(x) = \sum_{n\neq 0} s_n z^n \quad . \end{aligned} \tag{3.85}$$

The holomorphic functions τ, $\widetilde{\tau}$, υ and $\widetilde{\upsilon}$ generate the leading terms of the conifold local complex deformations. The expansion (3.55) describing the full set of local complex deformations of T^*S^3 can be thought of as,

$$T\,(x, y, z, w) = \tau\,(x) + \widetilde{\tau}\,(y) + \upsilon\,(z) + \widetilde{\upsilon}\,(w) + \mathcal{O}\,[higher] \quad , \tag{3.86}$$

where the higher terms are given by couplings generated by monomials type

$$[\tau\,(x)]^{m_1}\,[\widetilde{\tau}\,(y)]^{m_2}\,[\upsilon\,(z)]^{m_3}\,[\widetilde{\upsilon}\,(w)]^{m_4} \quad , \tag{3.87}$$

with m_i positive integers. The simplest non linear terms are given by the bilinears

$$\tau(x)\widetilde{\tau}(y) = \sum_{n,m>0} t_n t_{-m} x^n y^n \quad , \tag{3.88}$$

and

$$\upsilon(z)\widetilde{\upsilon}(w) = \sum_{n,m>0} s_n s_{-m} z^n w^n \quad . \tag{3.89}$$

More details on conifold local complex deformations and applications for 2D $c=1$ non critical string and topological string B model will given in the harmonic formulation we want to study now.

4 Conifold in harmonic framework

In this section, we develop the harmonic set up of conifold complex deformation analysis as it is the appropriate formalism in which $SL(2,C)$ isometry of the conifold is manifest. This formalism applies as well to the study of local deformations of S^3 preserving manifestly its $SU(2,C)$ symmetry of S^3 and to local complex deformations of cotangeant bundle T^*P^1 of complex one dimensional projective space $P^1 \sim S^2$.

To that purpose, we first describe harmonic variables as generally used in $4D$ $\mathcal{N}=2$ supersymmetric theories. Then, we use the covariant harmonic analysis to study local complex deformations of conifold and some specific submanifolds. After that, we give the full classification of these deformations in terms of $SU(2,C)$ and $SL(2,C)$ representations and establish a dictionary giving a *1 to 1 correspondence* between Fourrier expansion on circle and harmonic analysis on 3-sphere.

4.1 Harmonic variables

Roughly speaking, harmonic variables $u_\alpha^\pm$, with u_α^- being the complex conjugate of $u^{+\alpha}$ ($u_\alpha^- = \overline{u^{+\alpha}}$) are commuting $SU(2,C)$ isospinors ($s=\frac{1}{2}$) carrying two pairs of indices namely $\alpha = 1,2$ and the charges $\pm$. The first index ($\alpha = 1,2$) is related to the usual basic $SU(2,C)$ spin projection $s_z = \pm\frac{1}{2}$ by the relation,

$$s_z = \alpha - \frac{3}{2} \quad , \tag{4.1}$$

while the second index ($\pm$) refers to the fundamental charges of the Cartan Weyl subgroup $U_C(1)$ of the $SU(2,C)$ group. Recall that as a Lie group, $SU(2)$ can be split as

$$SU(2) = U_C(1) \times [SU(2)/U_C(1)]) \quad . \tag{4.2}$$

The $u^{\pm}_{\alpha}$ variables give a simple way to parameterize functions living on the *unit* real 3-sphere including its defining equations which read as,

$$\begin{aligned} u^{+\gamma}u^{-}_{\gamma} &= 1 \quad , \\ u^{+\gamma}u^{+}_{\gamma} &= 0 \quad , \\ u^{-}_{\gamma}u^{-\gamma} &= 0 \quad , \end{aligned} \tag{4.3}$$

where $f^{\gamma}g_{\gamma} = -f_{\gamma}g^{\gamma}$ stands for $\varepsilon_{\gamma\delta}f^{\gamma}g^{\delta}$ and will be often set as $f.g$. This definition is useful for studies involving harmonic expansion of function living on the sphere.
Local development may be also considered in harmonic frame work; all one has to do is to single out a given point on the sphere, say $u^{\pm}_{\alpha} = a^{\pm}_{\alpha}$ satisfying eqs(4.3), and use the identity

$$\varepsilon_{\alpha\beta} = \left(u^{+}_{\alpha}u^{-}_{\beta} - u^{-}_{\alpha}u^{+}_{\beta}\right) \quad , \tag{4.4}$$

to rewrite previous equations as follows,

$$\left(a^{+}.u^{+}\right)\left(a^{-}.u^{-}\right) - \left(a^{+}.u^{-}\right)\left(a^{-}.u^{+}\right) = 1 \quad . \tag{4.5}$$

For $u^{\pm}_{\alpha} = a^{\pm}_{\alpha}$, we recover $a^{+}.a^{+} = a^{-}.a^{-} = 0$ and $a^{+}.a^{-} = -a^{-}.a^{+} = 1$.
The use of $u^{\pm}_{\alpha}$ variables allows to avoid the complexity of $SU(2)$ tensor calculus and keeps $SU(2)$ symmetry manifest. In harmonic setting, $SU(2)$ irreducible tensors

$$T^{(\alpha_1 \dots \alpha_n)} \quad , \tag{4.6}$$

are described by the highest weight function,

$$T^{+n} = u^{+}_{(\alpha_1}\dots u^{+}_{\alpha_n)}T^{(\alpha_1\dots\alpha_n)} \quad , \tag{4.7}$$

carrying n Cartan charges and satisfying the $SU(2)$ highest weight state condition

$$\begin{aligned} D^{0}T^{+n} &= nT^{+n} \quad , \\ D^{++}T^{+n} &= 0 \quad . \end{aligned} \tag{4.8}$$

Local properties of harmonic functions are captured by the harmonic distributions

$$\frac{1}{\left(a^{+}.u^{+}\right)^{n}} \quad , \qquad n > 0 \quad , \tag{4.9}$$

and

$$\frac{1}{\left(a^{-}.u^{-}\right)^{n}} \quad , \qquad n > 0 \quad , \tag{4.10}$$

with pole singularities at $a^{+} = u^{+}$ and $a^{-} = u^{-}$ respectively. Cauchy's theorem of complex analysis reads in terms of harmonic variables as,

$$\int_{\gamma_a} \frac{a^{+}.du^{+}}{a^{+}.u^{+}} \sim 1 \quad , \tag{4.11}$$

where γ_a is a real contour surrounding the pole $a^+ = u^+$;

$$a^+ . u^+|_{a^+=u^+} = 0 \quad . \tag{4.12}$$

These features allow remarkable applications of harmonic variables in mathematical physics. Recall in passing that harmonic variables have been used to solve several problems in classical and quantum field theory requiring covariant $SU(2)$ tensor calculus.

Notice also that, in addition to the usual complex conjugation denoted here as $\left(\overline{X}\right)$, harmonic variables involve an extra conjugation $(*)$ reversing the sign of the charges of the $U(1)$ Cartan subgroup of $SU(2)$ symmetry. In the following table, we collect the general features of the conjugations $(-)$ and $(*)$ as well as their combination $(\overline{*})$ which for convenience we denote it as $(\sim)$.

field variables X	$\left(\overline{X}\right)$	(X^*)	$\overline{X}^* \equiv \widetilde{X}$
$w^{+\alpha}$	w_α^-	$w^{-\alpha}$	w_α^+
w_α^-	$w^{+\alpha}$	$-w^{-\alpha}$	$-w_\alpha^+$
$F^q(u^+, v^-)$	$\overline{F}^{-q}(u^-, v^+)$	$F^{-q}(u^-, v^+)$	$\widetilde{F^q}(u^+, v^-)$
$\widetilde{F^q}(u^+, v^-)$	$F^{-q}(u^-, v^+)$	$\overline{F}^{-q}(u^-, v^+)$	$(-)^q F^q(u^+, v^-)$

(4.13)

As one sees, $(\sim)$ is not a standard conjugation since $(\overline{*})^2 \neq (\overline{*})$. Like in 4D $\mathcal{N} = 2$ SYM, reality condition will be taken here also in the sense of $(\sim)$. Note finally that only harmonic function $F^q = F^q(u)$ that carry an even number of Cartan charges $(q = 2n)$ that can be subject to reality condition. Harmonic functions F^{2n+1} with an odd integer number of charge are necessary complex.

4.1.1 Harmonic variables and $4D$ $\mathcal{N} = 2$ supersymmetric theories

To our knowledge, harmonic variables have been first introduced for solving the problem of a manifestly superspace formulation of $4D$ $\mathcal{N} = 2$ extended Super Yang Mills [34] and $4D$ $\mathcal{N} = 2$ supergravity theories [38]-[40]. This method has been extended as well to their reductions down to $2D$ $\mathcal{N} = 4$ supersymmetric theories [54]-[58]. Then, they have been used for approaching different matters; in particular for studying $4D$ euclidean Yang Mills and gravitational instantons [42]-[47] and recently in dealing with the analysis of singularities of so called "hyperKahler" Calabi-Yau manifolds [43].
In the harmonic superspace formulation of $4D$ $\mathcal{N} = 2$ extended Super Yang Mills, the ordinary superspace with $SU(2)$ R-symmetry,

$$z^M = \left(x^\mu, \theta_a^\alpha, \overline{\theta}_{\overline{a}}^\alpha\right) \quad , \qquad \alpha = 1, 2 \quad , \tag{4.14}$$

gets mapped into the harmonic superspace

$$z^M = \left(Y^m, \theta_a^-, \overline{\theta}_{\overline{a}}^-, u_\alpha^\pm\right) \quad , \tag{4.15}$$

with

$$Y^m = \left(y^\mu, \theta_a^+, \overline{\theta}_{\overline{a}}^+\right) \quad , \tag{4.16}$$

and

$$\begin{aligned} y^\mu &= x^\mu + i\left(\theta^+ \sigma^\mu \overline{\theta}^- + \theta^- \sigma^\mu \overline{\theta}^+\right) \quad , \\ \theta_a^+ &= u_\alpha^+ \theta_a^\alpha, \qquad \overline{\theta}_{\overline{a}}^+ = u_\alpha^+ \overline{\theta}_{\overline{a}}^\alpha \quad . \end{aligned} \tag{4.17}$$

$4D$ $\mathcal{N} = 2$ matter superfields described by hypermultiplets are nicely represented in harmonic superspace. The typical Fayet-Iliopoulos hypermultiplet is described by a harmonic superfunction $Q^+(Y^m)$ carrying one positive Cartan-Weyl charge and has the following θ-expansion,

$$Q^+\left(y,\theta^+, \overline{\theta}^+, u\right) = u_\alpha^+ q^\alpha (y,u) + \theta^{+a} \psi_a (y,u) + \overline{\theta}_{\overline{a}}^+ \chi^{\overline{a}} (y,u) + ... \quad . \tag{4.18}$$

The gauge multiplet is described by the prepotential $V^{++}\left(y,\theta^+, \overline{\theta}^+\right)$ carrying two positive Cartan-Weyl charge; it reads in the Wess-Zumino gauge as

$$V^{++}\left(y,\theta^+, \overline{\theta}^+\right) = \theta^+ \sigma^\mu \overline{\theta}^+ \mathcal{A}_\mu (x) + \overline{\theta}^{+2} \theta^{+a} \lambda_a (x) + \theta^{+2} \overline{\theta}_{\overline{a}}^+ \lambda^{\overline{a}} (x) + \theta^{+2} \overline{\theta}^{+2} \mathcal{D}(x) \quad , \tag{4.19}$$

and it it used to covariantize the harmonic derivatives

$$D^{++} = u^+ \frac{\partial}{\partial u^-} \quad , \qquad \widetilde{D^{++}} = D^{++} \quad , \tag{4.20}$$

which becomes then

$$\mathcal{D}^{++} = D^{++} + V^{++} \quad , \qquad \widetilde{V^{++}} = V^{++} \quad . \tag{4.21}$$

With these tools, one can go ahead and write down the superfield action $\mathcal{S} = \mathcal{S}\left[Q^+, V^{++}\right]$ for $\mathcal{N} = 2$ SYM$_4$

$$\mathcal{S} = \int d^4x d^2\theta^+ d^2\overline{\theta}^+ \mathcal{L}^{4+}\left[Q^+, \widetilde{Q}^+, V^{++}\right] \quad . \tag{4.22}$$

The harmonic superspace lagrangian density $\mathcal{L}^{4+}\left[Q^+, \widetilde{Q}^+, V^{++}\right]$ carries four positive Cartan charges. The structure of its matter superfield dependence is,

$$\mathcal{L}^{4+} \sim \widetilde{Q}^+ \mathcal{D}^{++} Q^+ + \mathcal{L}_{int}^{4+} \quad , \tag{4.23}$$

with $\mathcal{L}_{int}^{4+}$ giving the matter self couplings; i.e $\mathcal{L}_{int}^{4+}\left(\widetilde{Q}^+, Q^+\right)$. For more details on eq(4.22) and its quantization, see [34, 40].

4.1.2 Harmonic analysis for conifold

The harmonic formalism for conifold is based on spinor representations of $SL(2,C)$ group; it goes beyond the $SU(2,C)$ harmonic analysis developed for $4D$, $\mathcal{N}=2$ supersymmetric gauge theories. The latter is recovered from conifold harmonic formalism as a special case by imposing reality condition and appropriate constraint eqs to be specified later.

Conifold harmonic analysis is used here to solve technical difficulties due to $SL(2,C)$ tensor calculus. Explicit examples will be given in forthcoming sections. It has moreover the advantage to shed more light on local complex deformations for conifold T^*S^3 and its subspace T^*P^1 by still keeping their isometries manifest.

To parameterize the conifold, we use the following set harmonic variables:

(**i**) A complex pair of variables

$$U^{+\gamma}=\left(U^{+1},U^{+2}\right)\quad , \tag{4.24}$$

transforming under $SU_u(2,C)$ as isospinor doublet $|s=\frac{1}{2},\quad s_z=\pm\frac{1}{2}>$; but as a degenerate one component state

$$|j=\frac{1}{2},\quad j_z=+\frac{1}{2}> \tag{4.25}$$

under the $SL(2,C)$ symmetry group. They are related to the previous small harmonic variables as

$$U^{+\gamma}=l\ u^{+\gamma}\quad . \tag{4.26}$$

(**ii**) A second complex pair of variables with opposite charge,

$$V_\gamma^-=\left(V_1^-,V_2^-\right)\quad , \tag{4.27}$$

transforming under $SU_v(2,C)$ as isospinor doublet; but under $SL(2,C)$ as a one component state as shown below,

$$|j=\frac{1}{2},\quad j_z=-\frac{1}{2}>\quad . \tag{4.28}$$

The vector (U^+,V^-) constitutes then an $SL(2,C)$ doublet of complex holomorphic variables. The two other partner pairs namely,

$$U_\gamma^-=\left(U_1^-,U_2^-\right)\quad ,\qquad V^{+\gamma}=\left(V^{+1},V^{+2}\right)\quad , \tag{4.29}$$

should be thought of as the anti-holomorphic variables. They are obtained from the previous ones by complex conjugation,

$$U_\gamma^-=\overline{U^{+\gamma}}\quad ,\qquad V^{+\gamma}=\overline{V_\gamma^-}\quad . \tag{4.30}$$

To fix the ideas, we recall that one should think of $U^{+\gamma}$ and V_γ^- as respectively associated with the complex holomorphic pairs (x, z) and (y, w),

$$U^{+\gamma} = (x, z) \quad , \qquad V_\gamma^- = (y, w) \quad . \tag{4.31}$$

The variables U_γ^- and $V^{+\gamma}$ correspond then to the anti-holomorphic partners $(\overline{x}, -\overline{z})$ and $(\overline{y}, -\overline{w})$,

$$U_\gamma^- = (\overline{x}, -\overline{z}) \quad , \qquad V^{+\gamma} = (\overline{y}, -\overline{w}) \quad . \tag{4.32}$$

Using this correspondence, and by help of the $SL(2, C)$ invariant metric $\varepsilon_{\alpha\beta}$ ($\varepsilon_{21} = -\varepsilon_{12} = 1$) for spinors, the conifold eq(3.31) can be easily shown to read as

$$U^{+\gamma}V_\gamma^- = \mu \quad , \tag{4.33}$$

$$U^{+\gamma}U_\gamma^+ = 0 \quad , \tag{4.34}$$

$$V^{-\gamma}V_\gamma^- = 0 \quad . \tag{4.35}$$

As one sees from these relations, the harmonic variables $U^{+\gamma}$ and V_γ^- scales as $\sqrt{\mu}$ that is like a lenght L. For convenience, it is interesting to use the rescaling

$$U^{+\gamma} = l_1 u^{+\gamma} \quad , \qquad V_\gamma^- = l_2 v_\gamma^- \quad , \qquad l_1 l_2 = \mu \quad , \tag{4.36}$$

where the small harmonic variables $u^{+\gamma}$ and v_γ^- are dimensionless and where l_1 and l_2 are two complex numbers whose module scale as length. This scaling property has the remarkable consequences.

(**a**) As far as singularities are concerned, the exact way to define the conifold is as

$$U^{+\gamma}V_\gamma^- = l_1 l_2 \quad , \tag{4.37}$$

where $\mu = 0$ is reached either by taking $l_1 = 0$, $l_2 \neq 0$ or $l_2 = 0$, $l_1 \neq 0$ or $l_1 = l_2 = 0$.

(**b**) Thinking about the ambient complex space C^4 as the product $C_u^2 \times C_v^2$ with

$$U^{\pm} \in C_u^2 \sim R_u^4 \qquad \text{and} \qquad V^{\mp} \in C_v^2 \sim R_v^4 \quad ,$$

one sees that the real 3-sphere is obtained by identifying V^- and U^- as shown below,

$$V^- = \left(\frac{l_1 l_2}{r^2}\right) U^- \quad , \tag{4.38}$$

where r is some real fixed number. In this case, the real slice of the conifold reads as follows,

$$U^{+\gamma}U_\gamma^- = r^2 \quad . \tag{4.39}$$

(**c**) The scaling (4.36) has also the effect of mapping the conifold real slice S^3,

$$U^{+\gamma}U_\gamma^- = \mathrm{Re}\,(\mu) = p \quad , \tag{4.40}$$

with radius $\sqrt{\mathrm{Re}\,(\mu)}$ to the unit sphere $u^{+\gamma}u_{\gamma}^{-}=1$. In general under the change (4.36), the generic conifold eqs(4.33-4.35) maps to,

$$\begin{aligned} u^{+\gamma}v_{\gamma}^{-} &= 1 \quad , \\ u^{+\gamma}u_{\gamma}^{+} &= 0 \quad , \\ v_{\alpha}^{-}v^{-\alpha} &= 0 \quad , \end{aligned} \tag{4.41}$$

where now the real slice is given bu the unit sphere.

4.2 Conifold isometries

As far as isometries of $T^{*}S^{3}$ are concerned, we have the following:
(**1**) The $SL_{u}\,(2,C)$ isometry subgroup factor[3] generated by the general coordinate change,

$$u^{+\gamma}\rightarrow\Lambda^{++}v^{-\gamma} \quad , \qquad v_{\gamma}^{-}\rightarrow v_{\gamma}^{-} \quad , \tag{4.42}$$

where Λ^{++} is priori an arbitrary complex function of u^{+} and v^{-}; i.e

$$\Lambda^{++}=\Lambda^{++}\left(u^{+},v^{-}\right) \quad . \tag{4.43}$$

The leading terms of the harmonic expansion of this complex holomorphic function reads, for the case of the charge is conserved, as

$$\Lambda^{++}=\Lambda_{(\alpha\beta)}u^{+\alpha}u^{+\beta}+\Lambda_{(\alpha_{1}\alpha_{2}\beta_{1}\beta_{2})}u^{+\alpha_{1}}u^{+\alpha_{2}}v^{-\beta_{1}}v^{-\beta_{2}}+\ldots \quad . \tag{4.44}$$

The general expression will be discussed later. Note that global $SL_{u}\,(2,C)$ invariance is described by the group parameter Λ_{0}^{++} constrained as

$$\frac{\partial\Lambda_{0}^{++}}{\partial v^{-}}=0 \quad , \qquad u^{+\alpha}\frac{\partial\Lambda_{0}^{++}}{\partial u^{+\alpha}}=2\Lambda_{0}^{++} \quad , \tag{4.45}$$

with harmonic expansion given,

$$\Lambda_{0}^{++}\left(u^{+},v^{-}\right)=\Lambda_{(\alpha\beta)}u^{+\alpha}u^{+\beta} \quad . \tag{4.46}$$

In this relation there is no v^{-} dependence and where one recognizes the $SL_{u}\,(2,C)$ triplet of complex parameters

$$\Lambda_{(\alpha\beta)}\equiv\left(\Lambda_{11},\Lambda_{(12)},\Lambda_{22}\right) \quad . \tag{4.47}$$

To make contact with the complex analysis of previous section, the harmonic transformations (4.42-4.46) should be associated with eqs(3.58).
(**2**) The $SL_{v}\,(2,C)$ isometry subgroup factor of the conifold reads as,

$$u^{+\gamma}\rightarrow u^{+\gamma} \quad , \qquad v_{\gamma}^{-}\rightarrow v_{\gamma}^{-}=\Gamma^{--}u_{\gamma}^{+} \quad , \tag{4.48}$$

[3] Here hermiticity is taken in the sense of the combined conjugation $\sim\equiv(\overline{*})$.

where, like before, the parameter Γ^{--} is an arbitrary holomorphic function in the harmonic variables u^{+} and v^{-}; i.e

$$\Gamma^{--} = \Gamma^{--}\left(u^{+}, v^{-}\right) \quad . \tag{4.49}$$

Its leading terms preserving manifestly global $SL_v(2, C)$ symmetry read as,

$$\Gamma_0^{--} = v_\alpha^- v_\beta^- \Gamma^{(\alpha\beta)} \quad , \qquad \frac{\partial \Gamma_0^{--}}{\partial u^{+\gamma}} = 0 \quad , \qquad -v^{-\alpha}\frac{\partial \Gamma_0^{--}}{\partial v^{-\alpha}} = -2\Gamma_0^{--} \quad , \tag{4.50}$$

In the language of section 2, this symmetry should be associated with the symmetry group generated by $\left\{\widetilde{L}_0, \widetilde{L}_\pm\right\}$, see also (3.58).

4.2.1 Generators

In eq(4.41) defining T^*S^3, there are two kinds of harmonic variables namely u^+ and v^-; but no complex conjugate u^- nor v^+. Each one of these holomorphic harmonic variables belongs to one of the two C^2 factors of the complex space $C^4 \sim C_u^2 \times C_v^2$. For later use, let us make a comment regarding these two sectors.

(**i**) In the u-sector, the $u^{+\gamma}$ holomorphic harmonic variable (belonging to C_u^2) together with its complex conjugate $u_\gamma^- = \overline{u^{+\gamma}}$ parameterize the real S^3 slice of T^*S^3,

$$U^{+\gamma}U_\gamma^- = \mathrm{p}u^{+\gamma}u_\gamma^- = \mathrm{p} \quad , \tag{4.51}$$

$$U^{+\gamma}U_\gamma^+ = \mathrm{p}u^{+\gamma}u_\gamma^+ = 0 \quad , \tag{4.52}$$

$$U^{-\gamma}U_\gamma^- = \mathrm{p}u^{-\gamma}u_\gamma^- = 0 \quad . \tag{4.53}$$

with $\mathrm{p} = (\mathrm{Re}\,\mu)$. Both capital harmonic variables $U_\alpha^\pm$; and the corresponding small $u_\alpha^\pm$ ones are rotated by the same $SU_u(2, C)$ algebra whose generators read as,

$$\begin{aligned} D_u^{++} &= u^{+\alpha}\frac{\partial}{\partial u^{-\alpha}} \quad , \\ D_u^{--} &= u^{-\alpha}\frac{\partial}{\partial u^{+\alpha}} \quad , \\ D_u^{0} &= u^{+\alpha}\frac{\partial}{\partial u^{+\alpha}} - u^{-\alpha}\frac{\partial}{\partial u^{-\alpha}} \quad , \end{aligned} \tag{4.54}$$

or equivalently, by using capital harmonic variables, as,

$$\begin{aligned} D_U^{++} &= U^{+\alpha}\frac{\partial}{\partial U^{-\alpha}} \equiv D_u^{++} \quad , \\ D_U^{--} &= U^{-\alpha}\frac{\partial}{\partial U^{+\alpha}} \equiv D_u^{--} \quad , \\ D_U^{0} &= U^{+\alpha}\frac{\partial}{\partial U^{+\alpha}} - U^{-\alpha}\frac{\partial}{\partial U^{-\alpha}} \equiv D_u^{0} \quad . \end{aligned} \tag{4.55}$$

The identification between the harmonic differential operators D_U^{++}, D_U^{--}, D_U^0 and D_u^{++}, D_u^{--}, D_u^0 respectively follows the fact that with the coordinate change $U^{\pm\alpha} = lu^{\pm\alpha}$, the scaling parameter l is an $SU_u(2,C)$ singlet which does not depend on harmonic variables. The identification is directly seen on the following typical tranformation,

$$D_u^{++} = \left(D_u^{++}U^{-\beta}\right)\frac{\partial}{\partial U^{-\beta}} + \left(D_u^{++}l\right)\frac{\partial}{\partial l} = D_U^{++} \quad . \tag{4.56}$$

The differential harmonic operators D_u^q, with $q = 0, \pm 2$, carry only Cartan-Weyl charges and no free $SU_u(2,C)$ spinor indices. Direct computations shows that they satisfy the usual $SU_u(2,C)$ commutation relations,

$$\begin{aligned} \left[D_u^{++}, D_u^{--}\right] &= D_u^0 \quad , \\ \left[D_u^0, D_u^{++}\right] &= 2D_u^{++} \quad , \\ \left[D_u^0, D_u^{--}\right] &= -2D_u^{--} \quad . \end{aligned} \tag{4.57}$$

Similar expressions are also valid for D_U^{++}, D_U^{--} and D_U^0; they show that $SU_u(2,C)$ symmetry remains invariant under global scaling of the harmonic variables. Acting by D_u^{++}, D_u^{--} and D_u^0 on eqs(4.51),

$$D_u^{++}\left(U^{+\gamma}U_\gamma^-\right) \quad , \tag{4.58}$$

and so on, one gets, on one hand, eq(4.52),

$$D_u^{++}\left(U^{+\gamma}U_\gamma^-\right) = \mathrm{p}u^{+\gamma}u_\gamma^+ = 0 \quad , \tag{4.59}$$

and on the other hand $D_u^{++}\mathrm{p}$. Consistency requires then

$$D_u^{++}\mathrm{p} = 0 \quad . \tag{4.60}$$

So the complex deformation parameter p should be an $SU_u(2,C)$ invariant. In general the relations defining this invariance read as,

$$D_u^{++}\mathrm{p} = D_u^{--}\mathrm{p} = D_u^0\mathrm{p} = 0 \quad . \tag{4.61}$$

This means also that $SU_u(2,C)$ rotations commute with the global volume fluctuations of the 3-sphere S^3. Since these deformations are generated by $\partial_p \equiv \frac{\partial}{\partial p}$; we then have,

$$\left[D_u^{++}, \frac{\partial}{\partial p}\right] = \left[D_u^{--}, \frac{\partial}{\partial p}\right] = \left[D_u^0, \frac{\partial}{\partial p}\right] = 0 \quad . \tag{4.62}$$

Details on variations generated by the local complex deformations beyond eq(4.61) will be given after making the following comment.

(**ii**) In the v-sector, the holomorphic harmonic variables v_γ^- belong to the second C^2 (C_v^2) copy of $C^4 \sim C_u^2 \times C_v^2$, where lives the conifold T^*S^3. Together with their complex conjugates $v^{+\gamma} = \overline{v_\gamma^-}$; these $v^\pm$ harmonic variables parameterize an other S^3 sphere which is actually embedded in C_v^2,

$$\begin{aligned} V^{+\gamma}V_\gamma^- &= \mathrm{q}v^{+\gamma}v_\gamma^- = \mathrm{q} \quad , \\ \varepsilon_{\gamma\delta}v^{+\gamma}v^{+\delta} &= 0, \qquad \varepsilon^{\gamma\delta}v_\alpha^- v_\beta^- = 0 \quad . \end{aligned} \tag{4.63}$$

Similarly as in u-sector, the capital harmonic variables $V_\alpha^\pm$ and the small $v_\alpha^\pm$ ones are rotated by the same $SU_v(2,C)$ algebra,

$$\begin{aligned} D_v^{++} &= v^{+\alpha}\frac{\partial}{\partial v^{-\alpha}} \quad , \\ D_v^{--} &= v^{-\alpha}\frac{\partial}{\partial v^{+\alpha}} \quad , \\ D_v^0 &= v^{+\alpha}\frac{\partial}{\partial v^{+\alpha}} - v^{-\alpha}\frac{\partial}{\partial v^{-\alpha}} \quad , \end{aligned} \tag{4.64}$$

satisfying the commutation relations,

$$\begin{aligned} \left[D_v^{++}, D_v^{--}\right] &= D_v^0 \quad , \\ \left[D_v^0, D_v^{++}\right] &= 2D_v^{++} \quad , \\ \left[D_v^0, D_v^{--}\right] &= -2D_v^{--} \quad . \end{aligned} \tag{4.65}$$

Like before, it is not difficult to check form eqs(4.63), that the complex deformation parameter q is an $SU_v(2,C)$ invariant,

$$D_v^{++}\mathrm{q} = D_v^{--}\mathrm{q} = D_v^0\mathrm{q} = 0 \quad . \tag{4.66}$$

Concerning the $SL(2,C)$ symmetry of eq(4.41) rotating the holomorphic $u^{+\gamma}$ and v_γ^- variables, we have the generators,

$$\begin{aligned} \nabla^{++} &= u^{+\alpha}\frac{\partial}{\partial v^{-\alpha}} \quad , \\ \nabla^{--} &= v^{-\alpha}\frac{\partial}{\partial u^{+\alpha}} \quad , \\ \nabla^0 &= u^{+\alpha}\frac{\partial}{\partial u^{+\alpha}} - v^{-\alpha}\frac{\partial}{\partial v^{-\alpha}} \quad , \end{aligned} \tag{4.67}$$

satisfying,

$$\begin{aligned} \left[\nabla^{++}, \nabla^{--}\right] &= \nabla^0 \quad , \\ \left[\nabla^0, \nabla^{++}\right] &= 2\nabla^{++} \quad , \\ \left[\nabla^0, \nabla^{--}\right] &= -2\nabla^{--} \quad . \end{aligned} \tag{4.68}$$

Similarly as before, we have here also the constraint eqs

$$\nabla^{++}\mu = \nabla^{--}\mu = \nabla^0\mu = 0 \quad , \tag{4.69}$$

showing that μ is invariant under the holomorphic $SL(2,C)$ symmetry. Moreover, global fluctuations of the conifold with respect to the variation of μ commute with $SL(2,C)$ rotations. So we have,

$$\left[\nabla^{++},\frac{\partial}{\partial\mu}\right]=\left[\nabla^{--},\frac{\partial}{\partial\mu}\right]=\left[\nabla^{0},\frac{\partial}{\partial\mu}\right]=0 \quad . \tag{4.70}$$

4.2.2 Antiholomorphic sector

Note in passing that along with ∇^{++}, ∇^{--} and ∇^{0}, we have also the anti-holomorphic differential operator given by,

$$\begin{aligned}\overline{\nabla}^{++} &= v^{+\alpha}\frac{\partial}{\partial u^{-\alpha}} \quad ,\\ \overline{\nabla}^{--} &= u^{-\alpha}\frac{\partial}{\partial v^{+\alpha}} \quad ,\\ \overline{\nabla}^{0} &= \left[\overline{\nabla}^{++},\overline{\nabla}^{--}\right] \quad .\end{aligned} \tag{4.71}$$

These operators which obey an $\overline{SL(2,C)}$ algebra; deals with the complex conjugate sector $SL(2,C)$; they don't concern us in present study since all the harmonic functions

$$F=F\left(u^{+},v^{-}\right) \quad , \tag{4.72}$$

we will encounter in what follows have no u^{-} and v^{+} dependence. They are then are annihilated by these operators,

$$\overline{\nabla}^{++}F=0 \quad , \qquad \overline{\nabla}^{--}F=0 \quad . \tag{4.73}$$

To summarize, the description of the conifold harmonic frame work involves the harmonic variables u_{α}^{+} and v_{α}^{-} rotated under $SL(2,C)$ symmetry; but no u_{α}^{-} and v_{α}^{+}. There are two sector for $SL(2,C)$ isometry in harmonic space; the first denoted $SL_{u}(2,C)$ with group parameter function Λ^{++} and the second denoted $SL_{v}(2,C)$ with group parameter function Γ^{--}. Under reality condition

$$\Lambda^{++}=\widetilde{\Lambda^{++}} \quad , \qquad \Gamma^{--}=\widetilde{\Gamma^{--}} \quad , \tag{4.74}$$

these groups reduce respectively to $SU_{u^{+}}(2,C)$ and $SU_{u^{-}}(2,C)$.

4.3 $T^{*}P^{1}$ as a submanifold of $T^{*}S^{3}$

Here we show that, like for the relation between the three sphere S^{3} and complex one dimension projective space $P^{1}\sim S^{2}$, we have quite similar correspondence between $T^{*}P^{1}$ and $T^{*}S^{3}$. Recall that the 3-sphere S^{3} may be thought of as a non trivial fibration of a circle S^{1} over S^{2},

$$S^{3}\sim S^{1}\propto S^{2} \quad . \tag{4.75}$$

In the same way, there is an analogous link between the cotangeant bundle of the two sphere T^*P^1 and conifold T^*S^3. More precisely, one may think about T^*S^3 as given by the fibration,

$$T^*S^3 \sim C^* \propto T^*P^1 \quad . \tag{4.76}$$

This correspondence can be made more precise in group theoretic language where the above fibrations take well known expressions. The fibration (4.75) corresponds to the factorisation of $SU(2,C)$ group in terms of its $U(1,C)$ abelian subgroup and the $\frac{SU(2,C)}{U(1,C)}$ coset subgroup as shown below,

$$SU(2,C) \sim U(1,C) \times \frac{SU(2,C)}{U(1,C)} \quad . \tag{4.77}$$

Similarly the fibration (4.76) corresponds to the following factorisation of the group $SL(2,C)$,

$$SL(2,C) \sim GL(1,C) \times \frac{SL(2,C)}{GL(1,C)} \quad . \tag{4.78}$$

With this correspondence in mind the derivation of cotangeant bundle T^*P^1 from the previous harmonic analysis of T^*S^3 becomes a simple matter. It is obtained by fixing $GL(1,C) \simeq C^*$ symmetry subgroup of conifold harmonic equations,

$$U^{+\alpha}V_\alpha^- = \mu \quad , \qquad U^{+\alpha}U_\alpha^+ = 0 \quad , \qquad V^{-\alpha}V_\alpha^- = 0 \quad . \tag{4.79}$$

Recall that the C^* action generating the projective transformations is given by,

$$U^{+\gamma} \to \lambda U^{+\gamma} \quad , \qquad V_\gamma^- \to \frac{1}{\lambda}V_\gamma^- \quad , \qquad \lambda \in C^* \quad . \tag{4.80}$$

In the harmonic analysis, the fixing of $GL(1,C)$ symmetry is achieved by making the identification,

$$U^{+\gamma} \equiv \lambda U^{+\gamma} \quad , \qquad V_\gamma^- \equiv \frac{1}{\lambda}V_\gamma^- \quad , \tag{4.81}$$

which corresponds to taking the harmonic variables $\left(U_1^+, U_2^+, V_1^-, V_2^-\right)$ in the weighted projective space $WP^3_{(+1,+1,-1,-1)}$. The complex two dimension holomorphic variety T^*P^1 embedded in $WP^3_{(+1,+1,-1,-1)}$ can be then defined as

$$T^*P^1 = T^*S^3/C^* \quad . \tag{4.82}$$

The complex holomorphic T^*P^1 geometry is a complex two dimension Calabi-Yau manifold embedded in $WP^3_{(+1,+1,-1,-1)}$.
In the harmonic frame work, the complex two dimension Calabi-Yau manifold T^*P^1 is defined, like for the conifold, by

$$U^{+\gamma}V_\gamma^- = \mu \quad , \qquad \varepsilon_{\gamma\delta}U^{+\gamma}U^{+\delta} = 0 \quad , \qquad \varepsilon^{\gamma\delta}V_\alpha^- V_\beta^- = 0 \quad , \tag{4.83}$$

but now with $\left(U_1^+, U_2^+, V_1^-, V_2^-\right)$ belonging to $WP^3_{(+1,+1,-1,-1)}$.
More generally, functions $F^q\left(U^+, V^-\right)$ living on T^*P^1 with q an integer should be covariant objects; they are harmonic functions carrying a well defined C^* charge q. This means that contrary to functions on conifold, functions $F^q\left(U^+, V^-\right)$ on T^*P^1 are homogeneous harmonic functions constrained as,

$$F^q\left(\lambda U^+, \frac{1}{\lambda}V^-\right) = \lambda^q \, \mathrm{F}^q\left(U^+, V^-\right) \quad . \tag{4.84}$$

This constraint equation can be also put in the following equivalent form,

$$\left[\nabla^0, F^q\left(u^+, v^-\right)\right] = qF^q\left(u^+, v^-\right) \quad , \tag{4.85}$$

where

$$\nabla^0 = \left(u^{+\alpha}\frac{\partial}{\partial u^{+\alpha}} - v^{-\alpha}\frac{\partial}{\partial v^{-\alpha}}\right) \quad , \tag{4.86}$$

is the charge operator of the $SL\left(2, C\right)$ isometry. Note that similar conclusions are also valid for the real slices S^2 of T^*P^1. In particular, the defining equation of the real two sphere S^2 which is just $S^3/U\left(1\right)$ is obtained from the harmonic equations of the three sphere,

$$u^{+\gamma}u^-_\gamma = 1 \quad , \qquad u^{+\gamma}u^+_\gamma = 0 \quad , \qquad u^-_\gamma u^{-\gamma} = 0 \quad , \tag{4.87}$$

by requiring moreover the identification

$$u^{+\gamma} \equiv e^{i\varphi}u^{+\gamma} \quad , \qquad u^-_\gamma = e^{-i\varphi}u^-_\gamma \quad , \tag{4.88}$$

with $\varphi \in [0, 2\pi]$. Harmonic functions on S^2 are then homogeneous functions obeying

$$F^q\left(e^{i\varphi}u^+, e^{-i\varphi}u^-\right) = e^{iq\varphi}F^q\left(u^+, u^-\right) \quad . \tag{4.89}$$

This property may be also stated as

$$\left[D^0_u, F^q\left(u^+, u^-\right)\right] = qF^q\left(u^+, u^-\right) \quad . \tag{4.90}$$

5 Harmonic expansion of complex deformations

So far we have considered global deformations of the conifold singularity $U^{+\gamma}V^-_\gamma = 0$ in harmonic space which becomes then $U^{+\gamma}V^-_\gamma = \mu$. In this section we consider the case of local complex deformations where μ get replaced by an arbitrary function

$$\xi = \xi\left(U^{+\gamma}, V^-_\gamma\right). \tag{5.1}$$

Then we study the classification of these deformations by using harmonic space. This manifestly covariant harmonic analysis will be used later for completing partial results

on S^3 quantum cosmology model of Gukov-Sarakin and Vafa [30]; in particular in the derivation of the three following:
(**i**) the manifestly $SL(2,C)$ invariant conifold partition function

$$\mathcal{Z}_{top} = \mathcal{Z}_{top}\left(T^* S^3\right), \tag{5.2}$$

(**ii**) the manifestly $SU(2,C)$ invariant of the Hartle-Hawking probability density

$$\varrho = \left|\Psi\left(S^3\right)\right|^2, \tag{5.3}$$

and (**iii**) quantum cosmology correlation functions of the fluctuation fields.
Recall that in the standard formulation, the classification of local complex deformations

$$\left\{t_n, \widetilde{t}_n;\ n > 0\right\} \tag{5.4}$$

of the conifold has been considered from different point of views; in particular in connection with the study of the ground ring of $c = 1$ non critical string [16] and in relation with the computation of the partition function $\mathcal{Z}_{top}\left(t, \widetilde{t}\right)$ of topological string B model on conifold [15].

To begin note that along with the global deformations generated by the variation of the global modulus μ, there are infinitely many local deformations of conifold. These transformations, which are no longer isometries of the conifold, are complex deformations captured by harmonic functions depending on the local coordinates $U^{+\gamma}$ and V_γ^- of the conifold. In harmonic framework, the set

$$\mathcal{J} = \left\{\xi : T^* S^3 \longrightarrow C\right\} \tag{5.5}$$

of conifold local complex deformations is an infinite set generated by arbitrary harmonic functions

$$\xi = \xi\left(U^{+\gamma}, V_\gamma^-\right), \qquad U^{+\gamma} V_\gamma^- = \mu. \tag{5.6}$$

Under these local deformations, eqs(4.33-4.41) get mapped to,

$$U^{+\gamma\prime} V_\gamma^{-\prime} = \mu + \xi\left(U^{+\gamma}, V_\gamma^-\right), \tag{5.7}$$

where the new harmonic variables,

$$U^{+\prime} = U^{+\prime}\left(U^+, V^-\right), \qquad V^{-\prime} = V^{-\prime}\left(U^+, V^-\right), \tag{5.8}$$

parameterize the locally deformed conifold. As noted earlier, the parameter μ may be usually absorbed in ξ as a zero mode as shown on $U^{+\gamma\prime} V_\gamma^{-\prime} = f$ with $f = \mu + \xi$. For later use, we shall keep however this splitting. By comparing the manifolds $U^{+\gamma} V_\gamma^- = \mu$ and

$U^{+\gamma\prime}V_{\gamma}^{-\prime}=f$, one sees that the deformation corresponds to varying the global complex parameter μ by local moduli and are generated by the scaling,

$$U^{+\prime}=\Lambda U^{+},\qquad =V^{-\prime}=\Gamma V^{-}, \tag{5.9}$$

where $\Lambda=\Lambda\left(U^{+},V^{-}\right)$ and $\Gamma=\Gamma\left(U^{+},V^{-}\right)$ are two harmonic functions living on conifold; they are related to the function ξ as

$$\Lambda\Gamma=\mu+\xi. \tag{5.10}$$

On the conifold real slice, we have $\Gamma=\widetilde{\Lambda}$ and so $\Lambda\widetilde{\Lambda}=p+\xi$ with the restriction $\widetilde{\xi}=\xi$.

5.1 Local deformations

First observe that the harmonic functions $\xi\left(U^{+\gamma},V_{\gamma}^{-}\right)$ scales like μ but are no longer invariant under the $SL\left(2,C\right)$ conifold isometry group. In harmonic differential operator language, this means that we have,

$$\left[\nabla^{++},\xi\left(U^{+},V^{-}\right)\right]\neq0,\qquad\left[\nabla^{--},\xi\left(U^{+},V^{-}\right)\right]\neq0 \tag{5.11}$$

and so

$$\left[\nabla^{0},\xi\left(U^{+},V^{-}\right)\right]\neq0, \tag{5.12}$$

where the differential operators ∇^{++}, ∇^{0} and ∇^{--} are as in eqs(4.67). To classify these local complex deformations, it is interesting to consider first local deformations of $T^{*}P^{1}$; then come back to those of $T^{*}S^{3}$.

5.1.1 Local deformations of $T^{*}P^{1}$

Local complex deformation of $T^{*}P^{1}$ are special deformations of the conifold; they are given by subset $\mathcal{J}_{0}^{(0)}$ of harmonic function $\zeta^{0}=\zeta^{0}\left(U^{+},V^{-}\right)$ invariant under projective transformations

$$\left(U^{+},V^{-}\right)\longrightarrow\left(\lambda U^{+},\frac{1}{\lambda}V^{-}\right). \tag{5.13}$$

Note that under above scaling, we have in general the following sections,

$$\mathcal{J}_{0}^{(q)}=\left\{\quad \xi|_{T^{*}P^{1}}\equiv\zeta^{q}\right\},\qquad q\in Z. \tag{5.14}$$

As we will see later, the conifold deformations $\xi\left(U^{+},V^{-}\right)$ have in general the following decomposition

$$\xi\left(U^{+},V^{-}\right)=\sum_{n=-\infty}^{\infty}z^{q}\zeta^{q}\left(U^{+},V^{-}\right),\qquad U^{+\gamma}V_{\gamma}^{-}=\mu, \tag{5.15}$$

where $\zeta^q(U^+,V^-)$ are homogeneous harmonic functions constrained as follows:

$$\zeta^q\left(\lambda U^{+\gamma},\frac{1}{\lambda}V_\gamma^-\right)=\lambda^q\zeta^q\left(U^{+\gamma},V_\gamma^-\right),\qquad q\in Z, \tag{5.16}$$

for any non zero complex number λ. Local complex deformations of T^*P^1 are then associated with the restriction,

$$\xi|_{T^*P^1}=\zeta^0\left(U^{+\gamma},V_\gamma^-\right). \tag{5.17}$$

In the $SL(2,C)$ harmonic differential operator language, the symmetry property

$$\zeta^0\left(\lambda U^{+\gamma},\frac{1}{\lambda}V_\gamma^-\right)=\zeta^0\left(U^{+\gamma},V_\gamma^-\right) \tag{5.18}$$

means that local complex deformations of T^*P^1 are constrained as follows,

$$\left[\nabla^0,\zeta^0\left(U^{+\gamma},V_\gamma^-\right)\right]=0, \tag{5.19}$$

but we still have,

$$\nabla^{++}\zeta^0\left(U^{+\gamma},V_\gamma^-\right)\neq 0,\qquad \nabla^{--}\zeta^0\left(U^{+\gamma},V_\gamma^-\right)\neq 0. \tag{5.20}$$

By comparing with the global parameter μ, one sees that the special local deformations $\zeta^0\left(U^{+\gamma},V_\gamma^-\right)$ carries the same C^* charge as μ eq(5.19); but have a non trivial dependence on the harmonic variables eqs(5.20). The general solution of the constraint eq(5.19),

$$\mu+\zeta^0\left(U^+,V^-\right)=\zeta_0^0+\sum_{n=1}^{\infty}\zeta^{(\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n)}U^+_{(\alpha_1}\ldots U^+_{\alpha_n}V^-_{\beta_1}\ldots V_{\beta_n)}, \tag{5.21}$$

where the zero mode $\zeta_0^0=\mu$ and where the non zero harmonic modes $\zeta^{(\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n)}$ are $SL(2,C)$ tensors given by,

$$\zeta^{(\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n)}=\int_{T^*P^1}U^{(+\alpha_1}\ldots U^{+\alpha_n}V^{-\beta_1}\ldots V^{-\beta_n)}\zeta^0\left(U^+,V^-\right). \tag{5.22}$$

In getting this relation, we have used the following harmonic space identity for T^*P^1,

$$\int_{T^*P^1}(U^+)^{(m}(V^-)^{n)}(U^+)_{(k}(V^-)_{l)}=\frac{(-1)^n m!n!}{(m+n+1)!}\delta^{(i_1}_{(j_1}\cdots\delta^{i_{m+n})}_{j_{k+l})}. \tag{5.23}$$

For more details on the properties of harmonic variables, harmonic distributions and harmonic integration on T^*P^1; see appendix section.

Note that in the harmonic expansion (5.21) and because of the conservation of C^* charge, harmonic variables $U^+_{\alpha_i}$ and $V^-_{\beta_j}$ are usually coupled and appear everywhere in pairs. Now, using the above features of harmonic variables, one may classify local complex deformation parameters

$$\zeta^{(\alpha_1\ldots\alpha_n,\beta_1\ldots\beta_n)},\qquad n\geq 1, \tag{5.24}$$

in terms of the parameters of $SDiff\left(T^{*}P^{1}\right)$, the group of area preserving diffeomorphisms of $T^{*}P^{1}$. Indeed restricting, in a first step, the set $\mathcal{J}_{0}$ of local complex deformations of $T^{*}P^{1}$ to the subset $\mathcal{J}_{0}^{(n)}$ generated by the following constraint eqs,

$$\begin{aligned} \left(\nabla^{++}\right)^{n+1}\zeta^{0} &= 0, & (5.25)\\ \nabla^{0}\zeta^{0} &= 0. & (5.26) \end{aligned}$$

where n is a positive integer, we get the two following:

(**1**) The cardinal of $\mathcal{J}_{0}^{(n)}$ is exactly equal to the dimension of the $SL\left(n+1,C\right)$ group.

$$order\left(\mathcal{J}_{0}^{(N)}\right)=\dim\left[SL\left(N+1,C\right)\right]=N^{2}-1. \tag{5.27}$$

To see this identity, it is enough to check it on the $n=1$ and $n=2$ leading terms. For $n=1$, the solution ${\zeta^{0}}_{(1)}$ of the constraint eqs (5.25-5.26) reads as

$${\zeta^{0}}_{(1)}\left(U^{+\gamma},V_{\gamma}^{-}\right)=\zeta^{(\alpha,\beta)}U_{(\alpha}^{+}V_{\beta)}^{-}, \tag{5.28}$$

and so we have,

$$\begin{aligned} order\left(\mathcal{J}_{0}^{(2)}\right) &= order\left\{\zeta^{(1,1)},\zeta^{(1,2)},\zeta^{(2,2)}\right\}\\ &= 3=\left(1+1\right)^{2}-1. & (5.29) \end{aligned}$$

For the case $n=2$, the solution ${\zeta^{0}}_{(2)}$ of the previous constraint eqs is given by

$$\zeta_{(2)}^{0}\left(U^{+\gamma},V_{\gamma}^{-}\right)=\zeta^{(\alpha,\beta)}U_{(\alpha}^{+}V_{\beta)}^{-}+\zeta^{(\alpha\gamma,\beta\delta)}U_{(\alpha}^{+}U_{\gamma}^{+}V_{\beta}^{-}V_{\delta)}^{-}, \tag{5.30}$$

together with

$$order\left(\mathcal{J}_{0}^{(3)}\right)=3+5=8=\left(2+1\right)^{2}-1. \tag{5.31}$$

More generally, we have for generic case $n=N$, the solution ${\zeta^{0}}_{(N)}$,

$$\zeta^{0}\left(U^{+},V^{-}\right)=\sum_{j=1}^{N}\zeta^{\left(\alpha_{1}...\alpha_{j}\beta_{1}...\beta_{j}\right)}U_{(\alpha_{1}}^{+}...U_{\alpha_{j}}^{+}V_{\beta_{1}}^{-}...V_{\beta_{j})}, \tag{5.32}$$

and the order of the set $\mathcal{J}_{0}^{(N)}$ is given by

$$order\left(\mathcal{J}_{0}^{(N)}\right)=\sum_{j=1}^{N}\left(2j+1\right) \tag{5.33}$$

which is also equal to $\left(N+1\right)^{2}-1$.

(**2**) Since the set $\mathcal{J}_{0}$ corresponds just to the limit $\mathcal{J}_{0}^{(+\infty)}$, we have then the correspondence $\mathcal{J}_{0}\sim SL\left(\infty,C\right)$ which is nothing but the volume preserving diffeomorphism group $SDiff\left(T^{*}P^{1}\right)$.

In the end of this discussion note that the constraint eqs (5.25-5.26) which may be rewritten as

$$\begin{aligned} \left[\nabla^{++}, \left(\nabla^{++}\right)^{n}\xi^{0}\right] &= 0, && (5.34)\\ \left[\nabla^{0}, \left(\nabla^{++}\right)^{n}\xi^{0}\right] &= 2n\left(\nabla^{++}\right)^{n}\xi^{0}, && (5.35)\end{aligned}$$

transform in spin s representation of $SL\left(2,C\right)$ and the solutions $\zeta^{0}_{(n-j)}$ with $1\leq j\leq n-1$ are just kernels of of the harmonic operator $\left(\nabla^{++}\right)^{n}$. In the following table, we give the representation group structure of the harmonic modes $\zeta^{(\alpha_1...\alpha_n,\beta_1...\beta_n)}$,

Parameters	spin $SU_u\left(2\right)$	spin $SU_v\left(2\right)$	spin $SL\left(2\right)$	C^{*} charge	scale dim
$U^{+\alpha}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1
$V^{-\alpha}$	0	$\frac{1}{2}$	$\frac{1}{2}$	-1	1
$\mu\equiv\zeta_{(0,0)}$	0	0	0	0	2
$\zeta^{(\alpha\beta)}\equiv\zeta_{(1,1)}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	-0
$\zeta_{(n,n)}$	$\frac{n}{2}$	$\frac{n}{2}$	n	0	$2-2n$

, (5.36)

with n an arbitrary positive integer and where we have set $\zeta^{(\alpha_1...\alpha_n,\beta_1...\beta_n)}\equiv\zeta_{(n,n)}$.

5.1.2 Local deformations fo $T^{*}S^{3}$

In the conifold case there is no constraint eqs type (5.16,5.19) and so the set eq(5.5) of local complex deformations of $T^{*}S^{3}$ is richer than in $T^{*}P^{1}$. For $T^{*}S^{3}$, there no projective symmetry restricting the harmonic function $\xi\left(U^{+},V^{-}\right)$. However, because of the fibration $T^{*}S^{3}\sim C^{*}\times T^{*}P^{1}$, we can think about

$$\xi\left(U^{+},V^{-}\right) \tag{5.37}$$

with U^{+} and V^{-} on conifold as

$$\zeta\left(z,U^{+\prime},V^{-\prime}\right) \tag{5.38}$$

where now z and $(U^{+\prime},V^{-\prime})$ are the local coordinates C^{*} and $T^{*}P^{1}$ respectively. In this parameterization, conifold is embedded in the non compact projective space $WP^{4}_{(-1,1,1,-1,-1)}$ and so we have the correspondence,

$$\xi\left(T^{*}S^{3}\right) \qquad\Longleftrightarrow\qquad \zeta\left(\frac{1}{\lambda}z,\lambda U^{+\prime},\frac{1}{\lambda}V^{-\prime}\right)=\zeta\left(z,U^{+\prime},V^{-\prime}\right). \tag{5.39}$$

Dropping the primes on $U^{+\prime}$ and $V^{-\prime}$; then expanding $\zeta\left(z,U^{+},V^{-}\right)$ in a Laurent series in power of z, we get the following,

$$\zeta\left(z,U^{+},V^{-}\right)=\sum_{q=-\infty}^{\infty}z^{q}\zeta^{q}\left(U^{+},V^{-}\right), \tag{5.40}$$

where $\zeta^{q}\left(U^{+},V^{-}\right)$ are harmonic functions (sections) on $T^{*}P^{1}$ given by,

$$\zeta^{q}\left(U^{+},V^{-}\right)=\int_{\gamma_{0}}\frac{dz}{2i\pi}z^{-q-1}\zeta\left(z,U^{+},V^{-}\right),\tag{5.41}$$

with γ_{0} a contour path surrounding the origin in the C^{*} plane. As one sees, the Laurent modes $\zeta^{q}\left(U^{+},V^{-}\right)$ transform under the projective transformation $z\rightarrow\frac{1}{\lambda}z$, $U^{+}\rightarrow\lambda U^{+}$ and $V^{-}\rightarrow\frac{1}{\lambda}V^{-}$ as,

$$\zeta^{q}\left(\lambda U^{+},\frac{1}{\lambda}V^{-}\right)=\lambda^{q}\zeta^{q}\left(U^{+},V^{-}\right),\qquad q\in Z,\tag{5.42}$$

in agreement with results on homogeneous functions $\zeta^{q}\left(U^{+},V^{-}\right)$ living on $T^{*}P^{1}$ and

$$\nabla^{0}\zeta^{q}\left(U^{+},V^{-}\right)=q\zeta^{q}\left(U^{+},V^{-}\right).\tag{5.43}$$

Besides the case $q=0$ which is associated with local complex deformations of $T^{*}P^{1}$, the solution of eq(5.43) depends on the sign of q. For q positive, say $q=+m$, the general solution is given by the following harmonic expansion,

$$\begin{aligned}\zeta^{m}\left(U^{+},V^{-}\right) &= \mu^{\frac{m}{2}}u^{+}_{(\alpha_{1}}...u^{+}_{\alpha_{m})}\zeta^{(\alpha_{1}...\alpha_{m})}\\ &\quad \mu^{\frac{m+2}{2}}u^{+}_{(\alpha_{1}}...u^{+}_{\alpha_{m+1}}v^{-}_{\beta_{1})}\zeta^{(\alpha_{1}...\alpha_{m}\alpha_{m+1},\beta_{1})}+...,\end{aligned}\tag{5.44}$$

where we have used the scaling (4.36) and where the $SL\left(2,C\right)$ representation structure of $\zeta^{\left(\alpha_{1}...\alpha_{m}\alpha_{m+j},\beta_{1}...\beta_{j}\right)}$ is as in previous table; see also table below.
Moreover setting the convention notation,

$$\zeta_{(m+j,j)}\equiv\zeta^{\left(\alpha_{1}...\alpha_{m+j},\beta_{1}...\beta_{j}\right)},\tag{5.45}$$

eq(5.44) reads as

$$\zeta^{m}\left(U^{+},V^{-}\right)=\sum_{j=0}^{\infty}\zeta_{(m+j,j)}U^{(+m+j}V^{-j)}.\tag{5.46}$$

For q a negative integer, say $q=-m$, we have the harmonic expansion,

$$\begin{aligned}\zeta^{-m}\left(U^{+},V^{-}\right) &= \mu^{\frac{m}{2}}v^{-}_{\beta_{1}}...v^{-}_{\beta_{m}}\zeta^{(\beta_{1}...\beta_{m})}\\ &\quad+\sum_{j\geq1}\mu^{\frac{m+2j}{2}}u^{+}_{\alpha_{1}}...u^{+}_{\alpha_{j}}v^{-}_{\beta_{1}}...v^{-}_{\beta_{m+j}}\zeta^{\left(\alpha_{1}...\alpha_{j},\beta_{1}...\beta_{m+j}\right)}.\end{aligned}\tag{5.47}$$

Similarly, setting $\zeta_{(j,m+j)}\equiv\zeta^{\left(\alpha_{1}...\alpha_{j},\beta_{1}...\beta_{m+j}\right)}$, the previous expansion of $\zeta^{-m}\left(U^{+},V^{-}\right)$ reads also as

$$\zeta^{-m}\left(U^{+},V^{-}\right)=\sum_{j=0}^{\infty}\zeta_{(j,m+j)}U^{(+j}V^{-m-j)}.\tag{5.48}$$

Putting these expansions altogether, we have the general result on the harmonic expansion of local complex deformations of conifold,

$$\begin{aligned}\zeta\left(z,U^{+},V^{-}\right) &= \sum_{j=0}^{\infty}\zeta_{(j,j)}U^{(+j}V^{-j)} \\ &+\sum_{q=1}^{\infty}z^{q}\left(\sum_{j=0}^{\infty}\zeta_{(q+j,j)}U^{(+q+j}V^{-j)}\right) \\ &+\sum_{q=1}^{\infty}z^{-q}\left(\sum_{j=0}^{\infty}\zeta_{(j,q+j)}U^{(+j}V^{-q-j)}\right),\end{aligned} \tag{5.49}$$

which may be put into the following formal condensed form,

$$\zeta\left(z,U^{+},V^{-}\right)=\sum_{q=-\infty}^{\infty}z^{q}\left(\sum_{j=0}^{\infty}\left[\zeta_{(q+j,j)}U^{(+q+j}V^{-j)}+\zeta_{(j,q+j)}U^{(+j}V^{-q-j)}\right]\right). \tag{5.50}$$

Therefore the local complex deformations of conifold are classified as follows

	spin s_u	spin s_v	spin s	C^* charge	Scale dim
$\zeta_{(n,m)}$	$\frac{n}{2}$	$\frac{m}{2}$	$\frac{n+m}{2}$	$n-m$	$2-(\lvert n\rvert+\lvert m\rvert)$

, (5.51)

where n and m are arbitrary positive integers and where we have set

$$\zeta^{\left(\alpha_1\ldots\alpha_j,\beta_1\ldots\beta_{m+j}\right)}\equiv\zeta_{(j,m+j)} \tag{5.52}$$

and

$$\zeta^{\left(\alpha_1\ldots\alpha_m\alpha_{m+j},\beta_1\ldots\beta_j\right)}\equiv\zeta_{(m+j,,j)}. \tag{5.53}$$

A noted in section 2, a particularly interesting subclass of conifold complex deformations are those associated with eqs(**??**-3.87). In the harmonic language, these deformations correspond to

$$\zeta\left(z,U^{+},V^{-}\right)\sim\zeta\left(z,U^{+}\right)+\widetilde{\zeta}\left(z,V^{-}\right), \tag{5.54}$$

with

$$\zeta\left(z,U^{+}\right)=\sum_{q\geq 1}z^{q}\zeta^{q}\left(U^{+}\right) \tag{5.55}$$

and

$$\widetilde{\zeta}\left(z,V^{-}\right)=\sum_{q\geq 1}z^{-q}\zeta^{-q}\left(V^{-}\right) \tag{5.56}$$

where $\zeta^{\pm q}$ obey similar relations as before. These expansions recall the usual Fourrier expansion on the circle namely

$$t\left(\theta\right)=\sum_{n>0}t_{n}e^{in\theta}+\sum_{n>0}t_{-n}e^{-in\theta}. \tag{5.57}$$

In fact there a *1 to 1* correspondence between Fourrier analysis on the unit circle S^1 and the harmonic expansion of the unit S^3 sphere. In what follows, we give a dictionary allowing the passage between the two formalisms.

5.2 $S^1 \longleftrightarrow S^3$ dictionary

One of the lessons one learns from above harmonic analysis on the 3-sphere and the one developed in the appendix, is that there is a one to one correspondence between Fourrier expansion on S^1 and harmonic analysis S^3. This analysis extends also T^*S^1 and T^*S^3,

$$T^*S^1 \qquad \longleftrightarrow \qquad T^*S^3. \tag{5.58}$$

In the following table we collect some relevant correspondences which will be used in forthcoming sections. Other useful relations will be given at proper times.

	$S^1 \sim U(1)$	$S^3 \sim S^1 \times S^2$
basis $\quad n \in N$	$b_n = e^{in\theta}$,	$e^{in\theta} b_n = e^{in\theta} u^{(+n)}$,
	$b_{-n} = e^{-in\theta}$	$e^{-in\theta} b_{-n} = e^{-in\theta} u^{(-n)}$,
charge operator	$Q = \frac{\partial}{i\partial\theta}$	$D_u^0 = \left(u^{+\alpha}\frac{\partial}{\partial u^{+\alpha}} - u^{-\beta}\frac{\partial}{\partial u^{-\beta}}\right)$
eigenvalue eqs	$Qb_{\pm n} = \pm n b_{\pm n}$	$D_u^0 b_{\pm n} = \pm n b_{\pm n}$
Expansion	$F(\theta) = \sum_{n\neq 0} F_n e^{in\theta}$	$\zeta(z, u^+) + \widetilde{\zeta}(z, v^-) =$
		$\sum_{n>0}\left(e^{in\theta} b_n \zeta_{(n,0)} + e^{-in\theta} b_{-n} \zeta_{(0,n)}\right)$
		$\zeta_{(n,0)} = \zeta_{(\alpha_1 \ldots \alpha_n)}, \quad \zeta_{(0,n)} = \zeta^{(\beta_1 \ldots \beta_n)}$
integral measure	$\int_{S^1} \frac{d\theta}{2\pi} = 1$	$\int_{S^3} d^3u = \int_{S^1} \frac{d\theta}{2\pi} \int_{S^2} d^2u = 1$
Distributions	$\delta_{n,m} = \int_{S^1} \frac{d\theta}{2\pi} b_n b_{-m}$	$\delta^{(n,m)} = \int_{S^2} d^2u b_n b_{-m}$
Modes, $n > 0$	$F_n = \int_{S^1} \frac{d\theta}{2\pi} e^{-in\theta} F(\theta)$	$\zeta_{(n,0)} = \int_{S^2} d^2u \zeta^{+n}(u) b_{-n}$,
	$F_{-n} = \int_{S^1} \frac{d\theta}{2\pi} e^{in\theta} F(\theta)$	$\zeta_{(0,n)} = \int_{S^2} d^2u \zeta^{-n}(u) b_n$
products	$\int_{S^1} \frac{d\theta}{2\pi} F^2(\theta)$	$\int_{S^3} d^3u \left[\zeta(z, u^+, v^-)\right]^2 =$
	$= \sum_n F_n F_{-n}$	$\sum_n \int_{S^2} d^2u \zeta^{+n} \zeta^{-n} = \sum_n \zeta_{(n,0)} \zeta_{(0,n)}$

(5.59)

where we have set,

$$\begin{aligned} u^{(+n)} &= u^{(+\alpha_1} \ldots u^{+\alpha_n)}, \\ u^{(-n)} &= u^-_{(\beta_1} \ldots u^-_{\beta_n)}. \end{aligned} \tag{5.60}$$

By scaling the variables, one can immediately write down the corresponding relations for real spheres with generic radii.

More details on the properties of harmonic analysis, differential and integral calculus as well as harmonic distributions may be found in [37, 59] and appendix section. In what follows, we shall use the results on the table and send to appendix for technical details.

6 Topological string on conifold

We start by reviewing the structure of the partition function $\mathcal{Z}_{top}$ of the B model topological string on conifold using complex holomorphic analysis. Then we reconsider the building of $\mathcal{Z}_{top}$; but this time using harmonic frame work. By comparing results for the restriction of T^*S^3 down to T^*S^1 and using dictionary eqs(5.59), we conjecture the general structure of $\mathcal{Z}_{top} = \exp\left(\mathcal{F}\left(\zeta, \widetilde{\zeta}\right)\right)$ preserving manifestly $SL(2,C)$ symmetry. For genus zero for instance, the leading terms of free energy reads as,

$$\begin{aligned} \mathcal{F}_0\left(\zeta, \widetilde{\zeta}\right) &= -\frac{1}{g_s^2} \sum_{n>0} \frac{\mu^{n-2}}{n} \int_{T^*S^2} \left(\zeta^n \widetilde{\zeta}^{-n}\right) \\ &+ \frac{1}{g_s^2} \sum_{n_1+n_2+n_3=0} p^{\frac{|n_1|+|n_2|+|n_3|-2}{2}-2} \left(\int_{T^*S^2} \zeta^{n_1}\zeta^{n_2}\zeta^{n_3}\right) \\ &+ ... \end{aligned} \tag{6.1}$$

where for positive integers $\zeta^{+n} = \zeta^{+n}(u^+)$ and $\zeta^{-n} = \zeta^{-n}(v^-)$.

6.1 $\mathcal{Z}_{top}$ Partition function

There is a nice correspondence between $SU(2) \times SU(2)$ symmetry classifying the deformation parameters of the conifold,

$$xy - zw = \mu + T(x, y, z, w) \quad , \tag{6.2}$$

and the $SU(2) \times SU(2)$ symmetry of the conformal theory $c = 1$ at the self dual radius. Following the study of [15, 30], we distinguish two basic sectors.
The first one concern restricting $T(x, y, z, w)$ to local complex deformations to,

$$T_{momentum}(x, y) = t(x, y) \quad , \tag{6.3}$$

with $t(x, y)$ taken as,

$$t(x, y) = \tau(x) + \widetilde{\tau}(y) + \ ... \quad . \tag{6.4}$$

Local deformations involve the x and y variables only; i.e setting

$$z = w = 0 \quad . \tag{6.5}$$

This restriction corresponds to turning on momentum modes in the $c = 1$ non critical string theory.
The second sector concern the local complex deformations $T(x, y, z, w)$ taken as

$$T_{winding}(z, w) = s(z, w) \quad , \tag{6.6}$$

where now,

$$s(z, w) = \sigma(z) + \widetilde{\sigma}(w) + \ ... \quad , \tag{6.7}$$

involving z and w variables only; i.e

$$x = y = 0 \quad . \tag{6.8}$$

This restriction corresponds to turning on winding modes.
Arbitrary complex deformations of the conifold generated by arbitrary $T = T(x, y, z, w)$ correspond then to turning all modes; that is momentum modes

$$\{\tau_n, \widetilde{\tau}_n\} \tag{6.9}$$

and winding ones

$$\{\sigma_n, \widetilde{\sigma}_n\}. \tag{6.10}$$

In this general case however, there is only partial results and so needs more investigation. The harmonic set up we will develop later give precious informations for this issue; but to make comparisons let us develop a little bit the usual Laurrent complex holomorphic analysis by focusing in a first step on momentum and winding modes sectors separately. Then explore the way to couple them. Once we do this, we turn back to harmonic analysis formalism.

In the case where complex deformations $T(x, y, z, w)$ are taken as $T_{momentum}(x, y)$, one is mainly restricting local complex deformations of T^*S^3 to the complex one dimension T^*S^1 region of the conifold,

$$xy = \mu, \qquad z = w = 0, \tag{6.11}$$

which is then deformed as,

$$xy = \mu + \tau(x) + \widetilde{\tau}(y) \ + \text{corrections} \quad . \tag{6.12}$$

On the circle S^1 describing the real slice of the cotangeant bundle T^*S^1 and obtained by setting

$$y = \overline{x}, \tag{6.13}$$

the previous deformed equation reduces to,

$$|x|^2 = p + \tau(x) + \overline{\tau(x)} + \text{corrections} \quad , \tag{6.14}$$

where $p = \mathrm{Re}(\mu)$ is the squared radius of the circle S^1.

A similar analysis may be done for the winding sector. Local complex deformations of T^*S^3 is restricted to a second complex one dimension T^*S^1 region of the conifold namely

$$zw = \mu, \qquad x = y = 0 \quad . \tag{6.15}$$

Local complex deformations of the corresponding real slice geometry reads as

$$|z|^2 = p + \sigma(z) + \overline{\sigma(z)} + \text{corrections} \quad . \tag{6.16}$$

where $w = \overline{z}$ and $x = y = 0$.

6.1.1 Geometric interpretation

To make contact with the parametrisation of S^3 analysis eqs(3.46-3.50), which we recall below for convenience,

$$\begin{aligned} x(\vartheta,\psi,\varphi) &= \sqrt{p}e^{\frac{i}{2}(\psi-\varphi)}\sin\frac{\vartheta}{2}, \qquad y(\vartheta,\psi,\varphi) = \sqrt{p}e^{\frac{-i}{2}(\psi-\varphi)}\sin\frac{\vartheta}{2}, \\ z(\vartheta,\psi,\varphi) &= -\sqrt{p}e^{\frac{i}{2}(\psi+\varphi)}\cos\frac{\vartheta}{2}, \qquad w(\vartheta,\psi,\varphi) = \sqrt{p}e^{\frac{-i}{2}(\psi+\varphi)}\cos\frac{\vartheta}{2}, \end{aligned} \tag{6.17}$$

the previous two restrictions of S^3 down to S^1 correspond to the following:
Either fixing the degrees of freedom ϑ, φ and ψ angles as,

$$\vartheta = \pi, \qquad z = 0, \quad w = 0 \quad , \tag{6.18}$$

for the first circle

$$x = \sqrt{p}e^{\frac{i}{2}(\psi-\varphi)}, \quad y = \sqrt{p}e^{\frac{-i}{2}(\psi-\varphi)} \quad . \tag{6.19}$$

Or by fixing ϑ like

$$\vartheta = 0, \qquad x = y = 0 \quad , \tag{6.20}$$

leading to the second circle

$$z = \sqrt{p}e^{\frac{i}{2}(\psi+\varphi)}, \quad w = \sqrt{p}e^{\frac{-i}{2}(\psi+\varphi)} \quad . \tag{6.21}$$

To get an explicit expression of $\mathcal{Z}_{top}(\tau,\widetilde{\tau})$, one uses a set of approximations. First thinks about $T_{momentum}(x,y)$ as given by the following perturbative development,

$$t(x,y) = \tau(x) + \widetilde{\tau}(y) + \mathcal{O}\left(\frac{\tau\widetilde{\tau}}{\mu}, \frac{\tau^2}{\mu}, \frac{\widetilde{\tau}^2}{\mu}\right), \tag{6.22}$$

where, at leading order, left and right moving modes are decoupled. Implementation of couplings requires going beyond the leading order of the perturbation; they correspond to non linear corrections in the field parameters $\tau(x)$ and $\widetilde{\tau}(y)$. The one holomorphic variable functions $\tau(x)$ and $\widetilde{\tau}(y)$, supposed smaller with respect to μ,

$$\tau(x) < \mu, \qquad \widetilde{\tau}(y) < \mu, \qquad \tau(x)\widetilde{\tau}(y) < \mu^2 \quad , \tag{6.23}$$

for any $x, y \in T^*S^1 \subset T^*S^3$, generate special infinitesimal local complex deformations of the conifold. They have the following Laurent expansions

$$\tau(x) = \sum_{n=1}^{\infty} t_n x^n, \qquad \widetilde{\tau}(y) = \sum_{n=1}^{\infty} t_{-n} y^n \quad . \tag{6.24}$$

On the real slice of T^*S^1, the complex variable y get identified with $\overline{x}$. Solving the reduced one dimensional geometry $|x|^2 = p$ as

$$x = \sqrt{p}e^{i\theta}, \qquad y = \sqrt{p}e^{-i\theta} \quad , \tag{6.25}$$

then substituting in

$$t\left(x,\overline{x}\right)=\tau\left(x\right)+\overline{\tau}\left(\overline{x}\right)+O\left(2\right)\quad, \tag{6.26}$$

the leading fluctuations that are linear in t_n and t_{-n} take the form,

$$t\left(p,\theta\right)=\sum_{n=1}^{\infty}p^{\frac{|n|}{2}}\left(t_n+t_{-n}\right)e^{in\theta}\quad, \tag{6.27}$$

where now $t_{-n}=\overline{t_n}$.

6.1.2 Partition function

The B model partition function $\mathcal{Z}_{top}\left(\tau,\widetilde{\tau}\right)$ on the conifold is a holomorphic a functional depending on the complex deformation moduli μ and t_{+n}, $\widetilde{t}_{+n}$; and reads as usual as

$$\mathcal{Z}_{top}\left(\tau,\widetilde{\tau}\right)=\exp\mathcal{F}\left(\tau,\widetilde{\tau}\right) \tag{6.28}$$

where $\mathcal{F}\left(\tau,\widetilde{\tau}\right)$ is the free energy. Because of the correspondence between B model topological string on conifold and $c=1$ non critical string, $\mathcal{F}\left(\tau,\widetilde{\tau}\right)\equiv\mathcal{F}\left(t\right)$ is given by the free energy $\mathcal{F}_{c=1}\left(t\right)$ and has the genus g expansion,

$$\mathcal{F}\left(t\right)=\left(\frac{\mu}{g_s}\right)^2\mathcal{F}_0\left(t\right)+\sum_{g=1}^{\infty}\left(\frac{g_s}{\mu}\right)^{2g-2}\mathcal{F}_g\left(t\right)\quad, \tag{6.29}$$

where g_s stands for the string coupling constant and where the genus g free energy component $\mathcal{F}_g\left(t\right)$ has the following structure [30],

$$\mathcal{F}_g\left(t\right)=\sum_{m\geq 2}P_g^m\left(n_i\right)\prod_{i=1}^{n}\left(\mu^{\frac{|n_i|}{2}-1}t_{n_i}\right)\quad. \tag{6.30}$$

In this relation, $P_g^m\left(n_i\right)$ is a polynomial in momenta n_i and has as degree $d=d\left(m,g\right)$ depending on m and g. Following [49, 30], we have, amongst others, the two useful informations on $P_g^m\left(n_i\right)$. First $d\left(2,g\right)=4g-1$ and second for n a positive definite integer,

$$P_{g=1}^{m=2}\left(n\right)=\frac{1}{24}\left(n-1\right)\left(n^2-n-1\right),\qquad g=0,1,...\quad. \tag{6.31}$$

Restricting the analysis to the leading genus term, the free energy $\mathcal{F}\left(t\right)$ reduces mainly to the genus zero factor

$$g_s^{-2}\mu^2\mathcal{F}_0\left(t\right) \tag{6.32}$$

which, in QFT$_4$ language, corresponds to particular F-terms in the effective action of 4D supersymmetric Yang Mills theory. Genus $g\geq 1$ components given by

$$\mathcal{F}_1\left(t\right)+O\left[\left(\frac{g_s}{\mu}\right)^2\right] \tag{6.33}$$

are understood as gravitational corrections.

In the super yang Mills approximation, the contribution of the genus zero term in the free energy series read as,

$$\mu^2 \mathcal{F}_0 (t) = -\sum_{n>0} \frac{\mu^n}{n} t_n t_{-n} + \frac{L_3 (t)}{3!} + \frac{L_4 (t)}{4!} + O\left(t^5\right), \tag{6.34}$$

where, for commodity, we have set

$$\begin{aligned} L_3 (t) &= \sum_{n_1+n_2+n_3=0} \mu^{\frac{|n_1|+|n_2|+|n_3|-2}{2}} t_{n_1} t_{n_2} t_{n_3}, \\ L_4 (t) &= \sum_{n_1+n_2+n_3+n_4=0} (1-\max\{|n_i|\}) \mu^{\frac{|n_1|+|n_2|+|n_3|+|n_4|-4}{2}} t_{n_1} t_{n_2} t_{n_3} t_{n_4}. \end{aligned} \tag{6.35}$$

Before going ahead let us make three remarks which turn out to be helpful when we consider the introduction of harmonic analysis to approach partition function $\mathcal{Z}_{top}$.

The first remark deals with rewriting free energy $\mathcal{F}(t)$ as an integral over the circle S^1.

$$\mathcal{F}[p,t] = \int_{S^1} \frac{d\theta}{2\pi} \mathcal{H}[t(p,\theta)], \tag{6.36}$$

where $\mathcal{H}[t(p,\theta)]$ is a free energy density on S^1.

The second remark concerns implementation of winding modes and the third one deals with how these deformations can be made $SU(2)$ covariant.

(1) Free energy density $\mathcal{H}$

Expansion $t(p,\theta)$ as

$$t(p,\theta) = \sum_{n\neq 0}^{\infty} f_n(p,\theta) \tag{6.37}$$

with $f_n(p,\theta) = p^{\frac{|n|}{2}} t_n e^{in\theta}$. Then using the scaling operator $p\frac{\partial}{\partial p}$, which acts on $f_n(p,\theta)$ as

$$p\frac{\partial}{\partial p} f_n = \frac{|n|}{2} f_n, \tag{6.38}$$

and the specific property of integrals on circle; in particular

$$\delta_{n,m} = \int_{S^1} \frac{d\theta}{2\pi} e^{i(n-m)\theta}, \tag{6.39}$$

one can re-express the energy libre $\mathcal{F}_0[p,t]$ as a functional integral over S^1. Indeed for genus zero terms, one can show that free energy $\mu^2\mathcal{F}_0[p,t]$ can be rewritten as

$$\int_{S^1} \mathcal{H}_0[T(p,\theta)] \tag{6.40}$$

with

$$\mathcal{H}_0[T(p,\theta)] = \int_{S^1} T(p,\theta) \frac{2}{D} T(p,\theta) + \frac{1}{p}\int_{S^1} T^3(p,\theta) + O\left(\frac{1}{p^2}T^4\right), \tag{6.41}$$

where $D = p\frac{\partial}{\partial p}$. The higher terms

$$O\left(\frac{1}{p^2}T^4\right) \tag{6.42}$$

are corrections suppressed by the perturbation condition $T < p$. This condition is naturally ensured for the case $p \to \infty$.

(2) Modes couplings

Local complex deformations of the conifold may be classified into specific subsets. Generally speaking, conifold local deformations using perturbation theory ideas should read as

$$xy - zw = \mu + LCD \tag{6.43}$$

where the LCD stands for "the full set of local complex deformations". In perturbation theory, LCD has the following structure,

$$\begin{aligned} LCD \quad = \quad & +\sum_{n>0} (t_n x^n + t_{-n} y^n + s_n z^n + s_{-n} w^n) \\ & + MMC \ + \ WMC \ + \ MWC. \end{aligned} \tag{6.44}$$

In this relation, the term MMC refers to "momentum modes couplings" and WMC to "winding modes couplings" and finally MWC refers to "momentum-winding modes interaction".

Observe that in the pure momentum sector ($z = w = 0$),

$$xy - zw = \mu + \sum_{n>0} (t_n x^n + t_{-n} y^n) + MMC, \tag{6.45}$$

local complex deformations

$$t_n x^n + t_{-n} y^n \tag{6.46}$$

as well as MMC couplings type

$$t_n t_{-m} x^n y^m \tag{6.47}$$

are all of them invariant under

$$x \longrightarrow x e^{2i\pi}, \qquad y \longrightarrow y e^{-2i\pi} \quad . \tag{6.48}$$

This is just the symmetry of the real slice ($y = \overline{x}$, $z = w = 0$) of T^*S^1 we have considered earlier.

For winding modes sector, we have quite similar relations; in particular

$$xy - zw = \mu + \sum_{n>0} (s_n z^n + s_{-n} w^n) + WMC \quad , \tag{6.49}$$

with leading winding mode couplings given by,

$$s_n s_{-m} z^n w^m \quad . \tag{6.50}$$

Along with these two sectors; that is momentum sector (M-sector) and winding sector (W-sector) there are also couplings involving both sectors.

(3) $SL(2)$ covariance

In a manifestly $SL(2,C)$ covariant formalism, the M-sector and W-sector should a priori be related under $SL(2,C)$ transformations. This means thal along with MMC and WMC, we should have moreover couplings involving both momentum and winding modes. To get all of these couplings, we proceed as follows:
First think about the zero order perturbation of the conifold as corresponding just to the global deformation generated by μ, that is global deformation of the conic singularity $xy - zw = 0$ leading then to $xy - zw = \mu$.
In the leading perturbation order in $\tau(x)$, $\widetilde{\tau}(y)$, $\sigma(x)$ and $\widetilde{\sigma}(y)$, conifold local deformations read as,

$$xy - zw = \mu + \sum_{n>0} \left(\upsilon_{(n,0)} + \upsilon_{(0,n)}\right), \tag{6.51}$$

where, for later use, we have set

$$\begin{aligned} \upsilon_{(n,0)} &= t_n x^n + s_n z^n, \\ \upsilon_{(0,n)} &= t_{-n} y^n + s_{-n} w^n. \end{aligned} \tag{6.52}$$

Then think about the two components $\upsilon_{(n,0)}$ and $\upsilon_{(0,n)}$ respectively as just the upper and down components of a $(n+1)$ components vector multiplet $\boldsymbol{\upsilon}_{\frac{n}{2}}$; that is,

$$\boldsymbol{\upsilon}_{\frac{n}{2}} = \begin{pmatrix} \upsilon_{(n,0)} \\ \upsilon_{(n-1,1)} \\ \dots \\ \upsilon_{(1,n-1)} \\ \upsilon_{(0,n)} \end{pmatrix}. \tag{6.53}$$

Clearly $\upsilon_{(n,0)}$ and $\upsilon_{(0,n)}$ are just the highest and lowest components of a spin $s = \frac{n}{2}$ multiplet of the group $SL(2,C)$ generated by the set $\{\mathcal{K}_0, \mathcal{K}_\pm\}$ given by the step operators $\mathcal{K}_\pm$,

$$\begin{aligned} \mathcal{K}_+ &= x\frac{\partial}{\partial w} - z\frac{\partial}{\partial y} \quad , \\ \mathcal{K}_- &= w\frac{\partial}{\partial x} - y\frac{\partial}{\partial z} \quad , \end{aligned} \tag{6.54}$$

together with the Cartan-Weyl charge operator $\mathcal{K}_0$,

$$\mathcal{K}_0=\left(x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}\right)-\left(y\frac{\partial}{\partial y}+w\frac{\partial}{\partial w}\right) \quad . \tag{6.55}$$

By direct computations, we have the following remarkable relations defining respectively highest and lowest weight representations of $SL\left(2,C\right)$,

$$\begin{aligned}\mathcal{K}_+\upsilon_{(n,0)} &= 0 \quad ,\\ \mathcal{K}_0\upsilon_{(n,0)} &= n\upsilon_{(n,0)} \quad ,\end{aligned} \tag{6.56}$$

and

$$\begin{aligned}\mathcal{K}_-\upsilon_{(0,n)} &= 0 \quad ,\\ \mathcal{K}_0\upsilon_{(0,n)} &= -n\upsilon_{(0,n)} \quad .\end{aligned} \tag{6.57}$$

To get the remaining $\upsilon_{(n-k,k)}$ components of the spin $s=\frac{n}{2}$ multiplet, one applies successively $\mathcal{K}_-$ on $\upsilon_{(n,0)}$ or $\mathcal{K}_+$ on $\upsilon_{(0,n)}$. For generic states

$$\upsilon_{(n-k,k)}\sim\mathcal{K}_-^k\upsilon_{(n,0)} \quad , \tag{6.58}$$

we have the following solution,

$$\upsilon_{(n-k,k)}\sim\left(t_{n,k}x^{n-k}w^k+(-)^k\,s_{n,k}z^{n-k}y^k\right), \qquad 0\leq k\leq n \quad , \tag{6.59}$$

where now the complex numbers $t_{n,k}$ and $s_{n,k}$ carry both momentum and winding modes. A manifestly $SL\left(2,C\right)$ covariant formulation of the infinitesimal local complex deformations of the conifold should be then as in eq(6.59). The leading perturbation order should involve a development in terms of $SL\left(2,C\right)$ representations in agreement with known results on $SDiff(S^3)$. Highest and lowest weight components $\upsilon_{(n,0)}$ of $SL\left(2,C\right)$ multiplets $\boldsymbol{\upsilon}_{\frac{n}{2}}$ are associated with linear infinitesimal deformations in t_n and s_n. Couplings involving both momentum and winding ones are described by the states $\upsilon_{(n-k,k)}$ with $1\leq k\leq n-1$. For the special case $n=1$, something special happens; there is no coupling between momentum and winding modes. This is a strong point in favor of harmonic analysis to be consider below. Higher perturbation orders describe non linear momentum modes couplings, non linear winding mode couplings and non linear momentum-winding modes interactions. Non zero connected amplitudes $<\mathcal{T}_{r_1}...\mathcal{T}_{r_k}>$ are defined as usual as,

$$<\mathcal{T}_{r_1}...\mathcal{T}_{r_k}>=\frac{\delta^k\mathcal{F}\left(t\right)}{\delta t_{r_1}...\delta t_{r_k}}|_{t=0}, \qquad \sum_{i=1}^k r_i=0, \tag{6.60}$$

where for the case of momentum modes the energy libre is as in eqs(6.29-6.35).

6.2 $\mathcal{Z}_{top}$ in harmonic frame work

We first show how harmonic variables $U^{+\gamma}$ and V_{γ}^{-} describe momentum and winding mode units. Then we give the manifestly invariant expression of the free energy

$$\mathcal{F}_{top} = \mathcal{F}_{top}\left(U^{+}, V^{-}\right) \tag{6.61}$$

of B model topological string on conifold.

6.2.1 Harmonic variables as unit momentum and winding modes

In the harmonic framework where the harmonic variables $U^{+\gamma}$ and V_{γ}^{-} form two $SL\left(2,C\right)$ isodoublets, momentum and winding modes get an interesting representation in terms of conifold isometry. Unit momentum modes $m = \pm 1$ and units of winding $\omega = \pm 1$ combine as

$$\begin{pmatrix} m=1 \\ \omega=1 \end{pmatrix}, \qquad \begin{pmatrix} m=-1 \\ \omega=-1 \end{pmatrix}, \tag{6.62}$$

and are respectively associated with the components of two fundamental isodoublets

$$U^{+\gamma}, \qquad V_{\gamma}^{-} \tag{6.63}$$

This feature is directly seen on these harmonic variables by using the complex coordinates (x, y, z, w),

$$\begin{pmatrix} U^{+1} \\ U^{+2} \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}, \qquad \begin{pmatrix} V_1^{-} \\ V_2^{-} \end{pmatrix} = \begin{pmatrix} y \\ w \end{pmatrix}. \tag{6.64}$$

The trivial (linear) monomials x and y are associated with momentum mode $|m| = 1$ and the monomials z and w with winding modes $|\omega| = 1$. As pointed out previously, this is a special property of a general feature of $SL\left(2,C\right)$ multiplets.
For the $SL\left(2,C\right)$ isotriplets, we have

$$U^{+(\alpha_1}U^{+\alpha_2)} = \begin{pmatrix} x^2 \\ xz \\ z^2 \end{pmatrix}, \qquad V_{(\beta_1}^{-}V_{\beta_2)}^{-} = \begin{pmatrix} y^2 \\ yw \\ w^2 \end{pmatrix}, \tag{6.65}$$

where the monomials x^2 and y^2 are associated with momentum modes $|m| = 2$ and z^2 and w^2 with the windings $|\omega| = 2$. The crossed terms

$$xz, \qquad \text{and} \qquad yw \tag{6.66}$$

have mode charges as,

$$\left(|m|, |\omega|\right) = (1, 1) \quad ; \tag{6.67}$$

they describe couplings between units of momenta and windings.

6.2.2 Expression of $\mathcal{Z}_{top}$ in harmonic analysis

Besides the physical property given above, the power of harmonic frame work in dealing with topological string partition function comes moreover from an other remarkable feature of conifold. With harmonic variable coordinates, all goes as if we are dealing with linear perturbation in T^*S^1.

Under local complex deformations of eq $U^{+\gamma}V_\gamma^- = \mu$, which gets then mapped to the three complex dimension manifold,

$$U^{+\gamma}V_\gamma^- = \mu + \xi\left(U^+, V^-\right) \quad , \tag{6.68}$$

the function $\xi\left(U^+, V^-\right)$ has a remarkable harmonic expansion recalling the one used in dealing with the subspace T^*S^1. Indeed using the fibration property T^*S^3 as C^* fibered over the base T^*P^1 and considering local coordinates

$$\left(\sigma, U^{+1}, U^{+2}, V_2^-, V_2^-\right) \tag{6.69}$$

of $WP^4_{(-1,1,1,-1,-1)}$, local complex deformations

$$\xi\left(U^+, V^-\right) \equiv \varphi\left(\sigma, U^+, V^-\right) \tag{6.70}$$

may first be expanded in Laurent series as,

$$\varphi\left(\sigma, U^+, V^-\right) = \zeta^0 + \sum_{n=1}^{\infty}\sigma^n\zeta^{+n} + \sum_{n=1}^{\infty}\sigma^{-n}\zeta^{-n} \quad , \tag{6.71}$$

where the Laurent modes

$$\zeta^{0,\pm n} = \zeta^{0,\pm n}\left(U^+, V^-\right) \tag{6.72}$$

are harmonic functions on T^*P^1 constrained as follows

$$\zeta^q\left(\lambda U^+, \frac{1}{\lambda}V^-\right) = \lambda^q\zeta^q\left(U^+, V^-\right), \qquad \lambda \in C^* \quad . \tag{6.73}$$

The expansion (6.71) should be compared with the right hand of eq(6.44),

$$\begin{aligned}\mu + LCD \;&=\; \mu + \sum_{n>0}\left(t_n x^n + t_{-n}y^n + s_n z^n + s_{-n}w^n\right)\\ &\quad + MMC \;+\; WMC \;+\; MWC \quad .\end{aligned} \tag{6.74}$$

The two relations describe exactly the same thing; but expressed in two different coordinate systems. The difference is that in eq(6.71), $SL\left(2, C\right)$ symmetry is manifest while it is not in eq(6.44). By analyzing eq(6.71) and eq(6.44), we can conjecture the the explicit expression of $\mathcal{Z}_{top}\left(\mu, \xi\right)$ preserving manifestly SL$(2, C)$ conifold isometry.

Theorem 1 : ***Expression of*** $\mathcal{Z}_{top}(\mu,\xi)$ ***preserving manifestly*** $SL(2)$ ***isometry***
Denoting by $\mathcal{Z}_{T^*S^1}$ *and* $\mathcal{Z}_{T^*S^3}$ *the two following:*
(i) $\mathcal{Z}_{T^*S^1} = \mathcal{Z}_{top}(\mu,t)$*, the partition function of B model topological string on the locally deformed conifold*

$$xy - zw = \mu + t(x,y) \quad , \qquad t(x,y) = T(x,y,z,w)|_{z=w=0} \quad , \tag{6.75}$$

where the $t(x,y)$ *deformations are restricted to* T^*S^1 *with* S^1 *being the large circle of* S^3*.*
(ii) $\mathcal{Z}_{T^*S^3} = \mathcal{Z}_{top}(\mu,\xi)$ *the harmonic space partition function of B model topological string of the manifestly* $SL(2,C)$ *covariant locally deformed conifold,*

$$U^{+\gamma}V_\gamma^- = \mu + \xi\left(U^+,V^-\right) \quad , \tag{6.76}$$

with local deformations generated by $\xi(U^+,V^-)$*,*
Then we have the following 1-1 correspondence,

$$\mathcal{Z}_{T^*S^3}(\mu,\xi) \qquad \longleftrightarrow \qquad \mathcal{Z}_{T^*S^1}(\mu,t) \quad . \tag{6.77}$$

In above eqs, $t(x,y)$ *lives on* T^*S^1 *and expands in general as*

$$\begin{aligned} t(x,y) &= \sum_{n,m=0}^{\infty} t_{n,m}x^n y^m \\ &= t_{0,0} + \sum_{n=1}^{\infty} t_{n,0}x^n + \sum_{m=1}^{\infty} t_{0,m}y^m + \sum_{n,m\geq 1}^{\infty} t_{n,m}x^n y^m \quad , \end{aligned} \tag{6.78}$$

while ξ *lives on* T^*S^3 *and is given by the harmonic expansion*

$$\xi = \zeta^0 + \sum_{n=1}^{\infty}\left(\sigma^n\zeta^{+n} + \sigma^{-n}\zeta^{-n}\right) \quad , \tag{6.79}$$

with $\zeta^{\pm n} = \zeta^{\pm n}(U^+,V^-)$ *are homogeneous functions of degree* $\pm n$ *living on* T^*S^2 *and whose harmonic developments read as*

$$\begin{aligned} \zeta^{+n}\left(U^+,V^-\right) &= \sum_{j=0}^{\infty} U^+_{(\alpha_1}..U^+_{\alpha_{j+n}}V^-_{\beta_1}..V^-_{\beta_j)}\zeta^{\left(\alpha_1...\alpha_{j+n}\beta_1...\beta_j\right)} \quad , \\ \zeta^{-n}\left(U^+,V^-\right) &= \sum_{j=0}^{\infty} U^+_{(\alpha_1}..U^+_{\alpha_j}V^-_{\beta_1}..V^-_{\beta_{j+n})}\zeta^{\left(\alpha_1...\alpha_j\beta_1...\beta_{j+n}\right)} \quad . \end{aligned} \tag{6.80}$$

$\zeta^0 = \zeta^0(U^+,V^-)$ *is obtained from above relations by setting* $n = 0$.

The conjectured relation (6.77) allows to derive the manifestly $SL(2,C)$ expression of the partition function of topological string B model on conifold. Higher order perturbation theory described by the terms MMC, WMC and MWC of eq(6.74) are trivially captured by the harmonic space function ξ. But what have become the usual difficulties of standard formalism? The answer is that they are still present; but they have taken an other form. A way to see it is to split $\xi = \varphi(\sigma, U^+, V^-)$ in basic blocks as follows,

$$\begin{aligned} \varphi\left(\sigma, U^+, V^-\right) &= \left[\zeta\left(\sigma, U^+\right) + \delta\eta\left(\sigma, U^+, V^-\right)\right] \\ &+ \left[\widetilde{\zeta}\left(\sigma, V^-\right) + \delta\widetilde{\eta}\left(\sigma, U^+, V^-\right)\right] \quad . \end{aligned} \tag{6.81}$$

Then use the fibration $T^*S^3 \simeq T^*S^1 \times T^*S^2$ to expand these functions; first in Laurent series in σ as,

$$\begin{aligned} \zeta\left(\sigma, U^+\right) &= \sum_{n>0} \sigma^n \zeta^{+n}\left(U^+\right) \quad , \\ \widetilde{\zeta}\left(\sigma, V^-\right) &= \sum_{n>0} \sigma^{-n} \zeta^{-n}\left(V^-\right) \quad , \\ \delta\eta\left(\sigma, U^+, V^-\right) &= \sum_{n>0} \sigma^n \delta\eta^{+n}\left(U^+, V^-\right) \quad , \\ \delta\widetilde{\eta}\left(\sigma, U^+, V^-\right) &= \sum_{n>0} \sigma^{-n} \delta\eta^{-n}\left(U^+, V^-\right). \end{aligned} \tag{6.82}$$

Then develop the Laurent modes in harmonic series on T^*S^2. We have for $\zeta^{+n}(U^+)$

$$\zeta^{+n}\left(U^+\right) = U^{+(\alpha_1} ... U^{+\alpha_n)} \zeta_{(\alpha_1 ... \alpha_n)} \tag{6.83}$$

which up on using the convention notations

$$\begin{aligned} U_\alpha^+ &= \sqrt{p} u_\alpha^+, \\ U^{(+n)} &= U^{+(\alpha_1} ... U^{+\alpha_n)} \\ u^{(+n)} &= u^{+(\alpha_1} ... u^{+\alpha_n)} \end{aligned} \tag{6.84}$$

and

$$\begin{aligned} U^{(+n)} &= p^{\frac{n}{2}} u^{(+n)} \\ \zeta_{(\alpha_1 ... \alpha_n)} &\equiv \zeta_{(n,0)}, \end{aligned} \tag{6.85}$$

it can be rewritten as

$$\zeta^{+n}\left(U^+\right) = p^{\frac{n}{2}} u^{(+n)} \zeta_{(n,0)}. \tag{6.86}$$

We also have,

$$\begin{aligned} \delta\eta^{+n}\left(U^+, V^-\right) &= \sum_{j=1}^{\infty} U_{(\alpha_1}^+ .. U_{\alpha_{j+n}}^+ V_{\beta_1}^- .. V_{\beta_j)}^- \; \delta\eta^{\left(\alpha_1 ... \alpha_{j+n} \beta_1 ... \beta_j\right)} \\ &= \sum_{j=1}^{\infty} U^{(+n+j} V^{-j)} \; \delta\eta_{(n+j,j)}, \end{aligned} \tag{6.87}$$

Similar relations may be written down for both for $\widetilde{\zeta}\left(\sigma, V^{-}\right)$ and $\delta\eta\left(\sigma, U^{+}, V^{-}\right)$ using first Laurent expansion on $T^{*}S^{1}$ and second harmonic development on $T^{*}S^{2}$. In particular we have

$$\begin{aligned} \zeta^{-n}\left(V^{-}\right) &= V^{-}_{(\beta_1}..V^{-}_{\beta_n)}\widetilde{\zeta}^{(\beta_1...\beta_n)} \\ &= p^{\frac{n}{2}}v^{(-n)}\widetilde{\zeta}_{(0,n)}. \end{aligned} \tag{6.88}$$

Note that $\zeta^{+n}\left(U^{+}\right)$ and $\zeta^{-n}\left(V^{-}\right)$ are the leading terms in eq(6.81); they involve only $SL\left(2,C\right)$ irreducible representations with spins $(s,0)=(n+1,0)$ and $(0,s)=(0,n+1)$ captured by the completely symmetric tensors $\zeta_{(\alpha_1...\alpha_n)}$ and $\widetilde{\zeta}^{(\beta_1...\beta_n)}$. The extra terms $\delta\eta^{+n}\left(U^{+},V^{-}\right)$ and $\delta\widetilde{\eta}^{-n}\left(U^{+},V^{-}\right)$ involve infinite towers of $SL\left(2,C\right)$ representations and are the sources of difficulties raised above.
Integration on $T^{*}S^{2}$ of monomials,

$$M^{(\mathrm{n,m})} = \prod_{n_i}\delta\eta^{+n_i}\left(U^{+},V^{-}\right)\prod_{m_j}\delta\widetilde{\eta}^{-m_j}\left(U^{+},V^{-}\right), \tag{6.89}$$

involves computing infinite traces of product of $SL\left(2,C\right)$ representations. This is a technical difficulty which may be overcome by using approximation methods.

6.2.3 Small local complex deformations

A way to deal with the local complex deformations of conifold,

$$U^{+\gamma}V^{-}_{\gamma} = \mu + \varphi\left(\sigma, U^{+}, V^{-}\right), \tag{6.90}$$

with $\varphi\left(\sigma,U^{+},V^{-}\right)=\varphi$ as in eq(6.81), is to think about φ as a small perturbation around the global deformation parameter μ. In this view the splitting

$$\varphi = \zeta + \widetilde{\zeta} + \delta\eta + \delta\widetilde{\eta} \tag{6.91}$$

may be interpreted as follows. The term

$$\zeta\left(\sigma,U^{+}\right) + \widetilde{\zeta}\left(\sigma,V^{-}\right) \tag{6.92}$$

is thought of as generating the leading perturbation order and

$$\delta\eta\left(\sigma,U^{+},V^{-}\right) + \delta\widetilde{\eta}\left(\sigma,U^{+},V^{-}\right) = \mathcal{O}\left[\zeta\widetilde{\zeta}, (\zeta)^2, \left(\widetilde{\zeta}\right)^2\right] \tag{6.93}$$

as corresponding to higher perturbation orders in ζ and $\widetilde{\zeta}$. As such eq(6.90) can be put into the following form,

$$\begin{aligned} U^{+\gamma}V^{-}_{\gamma} &= \mu + \sum_{n>0} p^{\frac{n}{2}}\left(\sigma^{n}u^{(+n)}\zeta_{(n,0)} + \sigma^{-n}v^{(-n)}\widetilde{\zeta}_{(0,n)}\right) \\ &\quad + \sum_{n,m>0} p^{\frac{n+m}{2}}\left(\sigma^{n-m}u^{(+n)}v^{(-m)}\zeta_{(n,0)}\widetilde{\zeta}_{(0,m)}\right) \\ &\quad + \text{higher corrections}. \end{aligned} \tag{6.94}$$

On the real slice of the conifold obtained by setting $|\sigma| = 1$ ($\sigma = e^{i\theta}$) and $V^- = U^- = \overline{U^+}$, the above eq takes the following reduced form,

$$\begin{aligned} U^{+\gamma}U^-_\gamma &= \mu + \sum_{n>0} p^{\frac{n}{2}} \left[e^{in\theta}u^{(+n)}\zeta_n + e^{-in\theta}u^{(-n)}\overline{\zeta}_{-n}\right] \\ &+ \sum_{n,m>0} p^{\frac{n+m}{2}} \left(e^{i(n-m)\theta}u^{(+n)}v^{(-m)}\zeta_n\overline{\zeta}_{-m}\right) \\ &+ \text{higher corrections,} \end{aligned} \tag{6.95}$$

where we have set

$$\zeta_n = \zeta_{(n,0)}, \qquad\qquad \overline{\zeta}_{-n} = \widetilde{\zeta}_{(0,n)}.$$

The first term may be also put in the form

$$\sum_{n\neq 0}^{\infty} p^{\frac{|n|}{2}} e^{in\theta} u^{(n)}\zeta_n \tag{6.96}$$

and should be compared to

$$t\left(p,\theta\right) = \sum_{n\neq 0}^{\infty} p^{\frac{|n|}{2}} t_n e^{in\theta} \tag{6.97}$$

describing local complex deformations of the large circle S^1 of the conifold. Therefore we have the $1 \longrightarrow (n+1)$ correspondence between the momentum mode $t_{\pm n}$ on the large circle S^1 of the 3-sphere and the modes $u^{(\pm n)}\zeta_{\pm n}$ on the full 3-sphere,

$$t_{\pm n} \qquad \longleftrightarrow \qquad u^{(\pm n)}\zeta_{\pm n}. \tag{6.98}$$

To each section $t_{\pm n}$ on the large circle S^1 of the 3-sphere with $U(1)$ charge $\pm n$, corresponds a homogeneous harmonic function $\zeta^{\pm n}$ on the two sphere S^2 with Cartan Weyl charges $\pm n$.

6.2.4 Free energy in harmonic set up

Using the correspondence eqs(6.77,6.98) as well as the dictionary we have given in section 4, one can write down the explicit expression of the manifestly $SU(2,C)$ conifold free energy $\mathcal{F}\left(\zeta,\overline{\zeta}\right)$. By help of table (5.59), the $SU(2,C)$ covariantization of the identities

$$\sum_{n>0}^{\infty} \frac{2p^n}{n} t_{-n}t_n \tag{6.99}$$

and

$$\sum_{n_1+n_2+n_3=0} p^{\frac{|n_1|+|n_2|+|n_3|}{2}-1} t_{n_1}t_{n_2}t_{n_3} \tag{6.100}$$

read respectively as

$$\sum_{n>0}^{\infty} \frac{2p^n}{n} \left(\int_{S^2} \zeta^{-n}\left(u^-\right)\zeta^n\left(u^+\right)\right), \tag{6.101}$$

and

$$\sum_{n_1+n_2+n_3=0} p^{\frac{|n_1|+|n_2|+|n_3|}{2}-1}\left(\int_{S^2}\zeta^{n_1}\zeta^{n_2}\zeta^{n_3}\right). \tag{6.102}$$

Using harmonic integration rule on the 3-sphere described in appendix, these quantities can be also rewritten as

$$\int_{S^3}\overline{\zeta}\left(\sigma,U^-\right)\frac{2}{\nabla^0}\zeta\left(\sigma,U^+\right), \tag{6.103}$$

and

$$\frac{1}{p}\int_{S^3}\left(\Upsilon\left(p,u^{\pm}\right)\right)^3. \tag{6.104}$$

The $SL(2,C)$ manifestly partition function for B model topological string on conifold restricted to real slice reads then as

$$\mathcal{Z}_{top}\left(\zeta,\overline{\zeta}\right)=\exp\mathcal{F}\left(\zeta,\overline{\zeta}\right).$$

The free energy $\mathcal{F}\left(\zeta,\overline{\zeta}\right)$ is obtained from $\mathcal{F}\left(t,\overline{t}\right)$ eq(6.29-6.35) by substituting everywhere

$$t_{\pm n}$$

moduli by the homogeneous harmonic function

$$u^{(\pm n)}\zeta_{\pm n}=\zeta^{\pm n}\left(u^{\pm}\right)$$

on the 2-sphere, eq(6.98). For instance, the genus zero contribution to the total free energy $\mathcal{F}\left(\zeta,\overline{\zeta}\right)$ namely $g_s^{-2}\mu^2\mathcal{F}_0\left(\zeta,\overline{\zeta}\right)$ reads as follows,

$$\mu^2\mathcal{F}_0\left(\zeta,\overline{\zeta}\right)=-\sum_{n>0}\frac{\mu^n}{n}\int_{S^2}\left(\zeta^n\overline{\zeta}^{-n}\right)+\int_{S^2}L\left(\zeta,\overline{\zeta}\right), \tag{6.105}$$

where the function $L\left(\zeta,\overline{\zeta}\right)$ is an $SU(2,C)$ invariant function involving specific monomials in ζ and $\overline{\zeta}$ and whose two leading terms may be read as

$$L\left(\zeta,\overline{\zeta}\right)\simeq\frac{L_3\left(\zeta,\overline{\zeta}\right)}{3!}+\frac{L_4\left(\zeta,\overline{\zeta}\right)}{4!}+O\left[\left(\zeta,\overline{\zeta}\right)^5\right] \tag{6.106}$$

with,

$$L_3\left(\zeta,\overline{\zeta}\right)=\sum_{n_1+n_2+n_3=0} p^{\frac{|n_1|+|n_2|+|n_3|-2}{2}}\left(\int_{S^2}\zeta^{n_1}(u)\,\zeta^{n_2}(u)\,\zeta^{n_3}(u)\right), \tag{6.107}$$

and, by setting setting $C_{n_1,n_2,n_3,n_4}=(1-\max\{(|n_1|,|n_2|,|n_3|,|n_4|)\})$, we have as well,

$$L_4\left(\zeta,\overline{\zeta}\right)=\sum_{n_1+n_2+n_3+n_4=0} C_{n_1,n_2,n_3,n_4}p^{\frac{|n_1|+|n_2|+|n_3|+|n_4|-4}{2}}\left(\int_{S^2}\zeta^{n_1}(u)\,\zeta^{n_2}(u)\,\zeta^{n_3}(u)\,\zeta^{n_4}(u)\right). \tag{6.108}$$

With the explicit expression of the free energy $\mathcal{F}\left[\zeta,\overline{\zeta}\right]$, one can compute the manifestly $SU\left(2,C\right)$ covariant correlation function on arbitrary points of S^3. Non zero connected amplitudes $< F^{n_1}...F^{n_r} >$ in harmonic framework are defined as usual as,

$$< F^{n_1}\left(u_1^{\pm}\right)...F^{n_r}\left(u_k^{\pm}\right) >= \frac{\delta^k\mathcal{F}\left[\zeta,\overline{\zeta}\right]}{\delta\zeta^{-n_1}\left(u_1\right)...\delta\zeta^{-n_k}\left(u_k\right)}|_{\zeta,\overline{\zeta}=0}. \tag{6.109}$$

This relation is non trivial except for the case where the Cartan-Weyl charges n_i add to zero; i.e

$$\sum_{i=1}^{k} n_i = 0. \tag{6.110}$$

This feature is require by $SU\left(2,C\right)$ invariance. Having these results at hand, we turn now to study correlations functions in the GSV quantum cosmology on S^3 [30].

7 Correlation functions in GSV quantum cosmology

After a brief review on the GSV S^3 quantum cosmological toy model, we compute the correlation functions of conformal fields

$$\Phi_i = \Phi\left(U_i^{\pm}\right) \tag{7.1}$$

at generic points $U_i^{\pm}$ on the 3-sphere

$$U_i^{+\alpha}U_{\alpha i}^{-} = p. \tag{7.2}$$

This computation, which involves both momenta and winding corrections, is done by using harmonic framework and convariantized Hartle-Hawking probability density $\varrho = |\Psi|^2$ preserving manifestly $SU\left(2,C\right)$ isometry of S^3.

7.1 The GSV model

In the GKV quantum cosmology toy model, the four dimensional space time representing the cosmological real world is identified with a number N of $D3$ branes wrapping the three sphere S^3; the real slice of the conifold. To motivate this construction, it is interesting to recall the following steps.

Start with a flux compactification of $10D$ type IIB superstring on $T^*S^3 \times S^2 \times S^1$ with the RR 5-form field strength flux,

$$F_5 = F_3 \wedge \omega_2 \quad , \tag{7.3}$$

threading through $T^*S^3 \times S^2$ with ω_2 being the unit volume 2-form on S^2 and F_3 a real 3-form on T^*S^3.

Then consider a symplectic basis $\{A^i, B_j\}$ of $H_3\left(T^* S^3\right)$ homology with intersection numbers

$$\begin{aligned} A^i \cap A^j &= 0, \qquad B_i \cap B_j = 0 \quad , \\ A^i \cap B_j &= \delta^i_j, \qquad i,j = 0,1,...,h^{2,1} \quad , \end{aligned} \tag{7.4}$$

with $h^{2,1} = h^{2,1}\left(T^* S^3\right)$ being the complex dimension of the moduli space of the complex deformations of conifold.

After; choose an integral basis $\{\mathbf{a}_i, \mathbf{b}^j\}$ of 3-cocycles in $H^3\left(T^* S^3, Z\right)$ cohomology to decompose the F_3 field strength as

$$F_3 = \sum_{i=0}^{h^{2,1}} \left(P^i \mathbf{a}_i + Q_i \mathbf{b}^i\right) \quad , \tag{7.5}$$

with P^i and Q_i are respectively the magnetic and electric fluxes. The moduli space of the complex deformations of conifold is parameterized by the periods of the holomorphic 3-form Ω on the 3-cycles A^i and B_j as shown below

$$X^i = \int_{A_i} \Omega \quad , \qquad \frac{\partial \mathcal{F}_0\left(X\right)}{\partial X^i} = \int_{B^i} \Omega \quad , \tag{7.6}$$

with $X^i = \left(\operatorname{Re} X^i\right) + i \left(\operatorname{Im} X^i\right)$ projective coordinates.

Then write down the Hartlee-Hawking wave function $\Psi\left(X, \overline{X}\right)$ associated with the conifold fluctuations. To do so one has to overcome the difficulty due to the fact that the projective X^i and $\overline{X^i}$ variables do not commute in the BPS minisuperspace [19, 21]. This non commutative behaviour is solved by restricting the moduli space to the subspace parameterized by

$$X^i = P^i + \frac{i}{\pi}\Phi^i, \qquad P^i = \operatorname{Re}\left(X^i\right), \tag{7.7}$$

where the attractor mechanism has been used to fix the conifold complex structure as $\operatorname{Re}\left(X^i\right) = P^i$ and the potential $\mathcal{F}_0$ in terms of electric charge as

$$\operatorname{Re}\left(C \frac{\partial \mathcal{F}_0\left(X\right)}{\partial X^i}\right) = Q_i. \tag{7.8}$$

For the complex structure μ for example, the attractor mechanism eqs $P^i = \operatorname{Re}\left(C X^i\right)$ and $Q_i = \operatorname{Re}\left(C \frac{\partial \mathcal{F}_0(X)}{\partial X^i}\right)$ lead to

$$\frac{2}{g_s}\left(\operatorname{Re}\mu\right) = N = P^0, \tag{7.9}$$

with $N >> 1$ so that $\operatorname{Re}\mu / g_s >> 1$. An overall rescaling of the charge P^0 by the inverse of the topological string coupling constant g_s ($P^0 \rightarrow \frac{1}{g_s} P^0$), we bring it to the form $\left(\operatorname{Re}\mu\right) = N$ and so,

$$\mu = N + \frac{i}{\pi}\Phi^0, \qquad \mathcal{F}_0\left(\mu\right) = \frac{i\mu}{\pi g_s} \ln\left(\frac{2\mu}{\Lambda g_s}\right), \tag{7.10}$$

In terms of the commutative coordinate Φ^i, the Hartlee-Hawking wave function reads therefore as follows,

$$\Psi_{(P^i,Q_j)}\left(\Phi^i\right) = \mathcal{Z}_{top}\left(X^0, ..., X^{h^{2,1}}\right) e^{\sum_{j=0}^{h^{2,1}} \frac{1}{2} Q_j \Phi^j}, \tag{7.11}$$

where $\mathcal{Z}_{top}(X)$ is the partition function of the B-model topological string on conifold. Next consider a conifold geometry in the limit of a large S^3 with a radius$\sqrt{p}$ greater than some given scale l_0 beyond which perturbation theory for higher genus corrections holds,

$$\sqrt{p} > l_0 = \Lambda\sqrt{g_s} \quad . \tag{7.12}$$

This condition follows from implementation of momentum modes contribution and the perturbative study of the higher genus corrections to the propagator

$$< t_n t_{-n} > \tag{7.13}$$

where it has been observed that the right perturbation parameter is

$$\frac{n^2 g_s}{\mu} \tag{7.14}$$

rather than $\frac{g_s}{\mu}$. The parameter Λ is a cut off parameter restricting local deformations $\sum_{n=-\infty}^{\infty} p^{\frac{|n|}{2}} t_n e^{in\theta}$ of the round 3-sphere into,

$$t(x,\overline{x}) = \sum_{n=-\Lambda}^{\Lambda} p^{\frac{|n|}{2}} t_n e^{in\theta} \quad . \tag{7.15}$$

The four dimensional cosmological real world is identified with a number N of $D3$ branes wrapping the three sphere S^3. Local fluctuations deform the shape of the three sphere and induce perturbations $\delta g_{\mu\nu}^{(0)}$ of round metric $g_{\mu\nu}^{(0)}$ which becomes then,

$$g_{\mu\nu}(t) = g_{\mu\nu}^{(0)} + \frac{\delta g_{\mu\nu}^{(0)}}{\delta t}\delta t + O\left(t^2\right). \tag{7.16}$$

Local complex deformations (7.15) deform the holomorphic form Ω_0 to $\Omega(t)$,

$$\Omega = f(t)\,\Omega_0 \tag{7.17}$$

where $f(t)$ is a scale factor. On the real slice of the conifold, these infinitesimal deformations induce fluctuations of the real part $\mathrm{Re}(\Omega_0)$. The new real part $\mathrm{Re}(\Omega)$ is related to $\mathrm{Re}(\Omega_0)$ as,

$$\frac{\mathrm{Re}\,\Omega}{\mathrm{Re}\Omega_0} = \exp\left[\Phi\left(t,\overline{t}\right)\right]. \tag{7.18}$$

where

$$\Phi = \Phi\left(t,\overline{t}\right) \tag{7.19}$$

is the conformal factor. It is this field $\Phi\left(t,\overline{t}\right)$ which is at central focus in GSV cosmology; it is assumed that its fluctuations would be observable. Local perturbations of the conifold, and so of the 3-sphere, should be felt on the D3 brane world as metric inhomogeneities and matter fluctuations. Note also that in this picture supersymmetry is supposed to be weakly broken so that mini-superspace approximation for stringy Hartle-Hawking wave function is still valid.

In what follows, we use harmonic set up considered in previous sections to compute the correlation functions of conformal factor in case where both momentum and winding corrections are implemented.

7.2 Harmonic differential geometry on T^*S^3

We begin by giving the explicit expression of the holomorphic volume of conifold in harmonic set up. Then we consider its real reduction on the three sphere and study field fluctuations.

7.2.1 Holomorphic 3-form in harmonic space

In the local coordinate system $\left(\sigma,U^{+1},U^{+2},V_1^-,V_2^-\right)$ of $WP^4_{(-1,+1,+1,-1,-1)}$ where conifold

$$\begin{aligned} U^{+\alpha}V_\alpha^- &= \mu, \qquad \sigma, \\ U^{+\alpha} &\equiv \lambda U^{+\alpha}, \qquad V_\alpha^- \equiv \frac{1}{\lambda}V_\alpha^-, \qquad \sigma\equiv\frac{1}{\lambda}\sigma, \end{aligned} \tag{7.20}$$

is embedded[4], the globally defined holomorphic 3-form Ω_3 of conifold is given by,

$$\Omega_3=\frac{d\sigma}{\sigma}dU^{+\alpha}dV^{-\beta}\varepsilon_{\alpha\beta}. \tag{7.21}$$

Clearly this holomorphic 3-form is $SL\left(2,C\right)$ invariant and its integration over a 3-cycle $A\sim\mathcal{C}_{\sigma=0}\times D$ of $H_3\left(T^*S^3,Z\right)$, with $\partial D\neq 0$, gives the complex structure μ; i.e

$$\mu\sim\int_A\Omega_3. \tag{7.22}$$

Indeed, a first integration of Ω_3 along the curve $\mathcal{C}_{\sigma=0}$ of σ-plane containing the origin $\sigma=0$ gives the reduced integral

$$\int_D\Omega_2 \tag{7.23}$$

of the holomorphic 2-form

$$\Omega_2=dU^{+\alpha}dV^{-\beta}\varepsilon_{\alpha\beta} \tag{7.24}$$

[4]Recall that in projective capital letters, conifold eq is given by $U^{+\alpha}U_\alpha^-=\mu$ and σ free. For simplicity we will also use projective small letters $u^{+\alpha}v_\alpha^-=1$. The passage between the two coordinates is $U^{+\alpha}=\sqrt{\mu}u^{+\alpha}$, $V_\alpha^-=\sqrt{\mu}v_\alpha^-$.

over the 2-cycle D. Thus we have,

$$\mu = \int_{D\times\mathcal{C}_{\sigma=0}} \Omega_3 = \int_D \Omega_2. \tag{7.25}$$

By Stokes theorem, one can bring the 2D integral to a 1D integration on the non zero boundary ∂D.

$$\int_{\partial D} \Omega_1 = \int_{\partial D} U^{+\alpha} dV^{-\beta} \varepsilon_{\alpha\beta}. \tag{7.26}$$

Next consider the curve $\mathcal{C}_{a^+,b^-}$ with the following features:

(**i**) $\mathcal{C}_{a^+,b^-}$ contains the particular conifold point

$$\left(U^{+\alpha}, V_\beta^-\right) = \left(a^{+\alpha}, b_\beta^-\right) \tag{7.27}$$

satisfying obviously the constraint eq

$$a^{+\alpha} b_\alpha^- = \mu \tag{7.28}$$

and surrounding the pole singularities

$$\frac{a^{+\alpha}}{a^+.U^+} \quad , \qquad \frac{b^{-\alpha}}{b^{-\alpha} V_\alpha^-} \quad , \tag{7.29}$$

with residue $\left(a^{+\alpha}, b_\beta^-\right)$.

(**ii**) solves locally the equation $U^{+\alpha} V_\alpha^- = \mu$ inside of $\mathcal{C}_{a^+,b^-}$; i.e in the vicinity of $\left(a^{+\alpha}, b_\beta^-\right)$, by expressing locally $U^{+\alpha}$ in terms of the inverse of V_α^-. We have

$$U^{+\alpha} = \mu \frac{b^{-\alpha}}{b^-.V^-}, \tag{7.30}$$

where we have set $b^-.V^- = b^{-\alpha} V_\alpha^-$. A similar treatment may be done also for $V^{-\alpha}$ in terms of the inverse of $U^{+\alpha}$. We have

$$V^{-\alpha} = \mu \frac{a^{+\alpha}}{a^+.U^+} \tag{7.31}$$

with $a^+.U^+ = a^{+\alpha} U_\alpha^+$. Now, if we take the boundary ∂D as given by the curve $\mathcal{C}_{a^+,b^-}$, we get

$$\int_{\mathcal{C}_{a^+,b^-}} U^{+\alpha} dV^{-\beta} \varepsilon_{\alpha\beta} = \mu \int_{\mathcal{C}_{a^+,b^-}} \frac{b^{-\alpha} dV^{-\beta}}{b^-.V^-} \varepsilon_{\alpha\beta}. \tag{7.32}$$

Since $a^{+\alpha}$ and b_β^- are fixed moduli ($da^{+\alpha} = db_\beta^- = 0$), the 1-form $b^{-\alpha} dV^{-\beta}$ read also as $d\upsilon$ with $\upsilon = (b^-.V^-)$ and so

$$\int_{\mathcal{C}_{a^+,b^-}} \frac{b^{-\alpha} dV^{-\beta}}{b^-.V^-} \varepsilon_{\alpha\beta} = \int_{\mathcal{C}_{a^+,b^-}} \frac{d\upsilon}{\upsilon} = 2i\pi. \tag{7.33}$$

We get at the end

$$\int_A \Omega_3 = 2i\pi\mu. \tag{7.34}$$

Note that the same result can be obtained by using the 1-form $V^{-\beta}dU^{+\alpha}\varepsilon_{\alpha\beta}$. In this case the corresponding one dimensional integral reads as

$$\int_{\mathcal{C}_{a^+,b^-}} \frac{a^{+\alpha}dU^{+\beta}}{a^+.U^+}\varepsilon_{\alpha\beta} = \int_{\mathcal{C}_{a^+,b^-}} \frac{d\left(a^+.U^+\right)}{a^+.U^+} \tag{7.35}$$

and has a pole singularity at $a^+ = U^+$.

7.2.2 Fluctuations

In quantum cosmology model on S^3, one is interested in the fluctuations of the real volume of the round 3-sphere which becomes then

$$\Omega = e^{\Phi}\Omega_0. \tag{7.36}$$

In this relation, the local scaling factor e^{Φ} is just the absolute value of the Jacobian $J = \left|\frac{\partial U^{\pm\prime}}{\partial U^{\pm}}\right|$ of the general coordinate transformations

$$U^{\pm} \to U^{\pm\prime} = U^{\pm\prime}\left(U^{\pm}\right) \tag{7.37}$$

mapping Ω_0 into Ω. These fluctuations, which on $c = 1$ non critical string side describe momenta and winding corrections, are generated by local perturbations of the parameter p deforming the shape of the 3-sphere. To implement such deformations, the previous round sphere analysis relying on $U^{+\gamma}U^{-}_{\gamma} = p$ should be now extended to the hypersurface

$$U^{\prime+\gamma}U^{\prime-}_{\gamma} = F\left(p\right) \tag{7.38}$$

with

$$p \to p' = F\left(p\right) = p - \xi\left(p, u^{\pm}\right), \tag{7.39}$$

where $\xi\left(p, u^{\pm}\right)$ is a real harmonic function on S^3 and $\left(p, u^{\pm}\right)$ are the local coordinates of $R^+ \times SU\left(2\right) \sim R^4 \sim C^2_u$. Note that since there is a zero mode ξ_0 inside $\xi\left(p, u^{\pm}\right)$ and seen that it is already singled out $(p = -\xi_0)$, the right way to write the deformation $\xi\left(p, u^{\pm}\right)$ would be,

$$\xi\left(p, u^{\pm}\right) = D^{++}\overline{\xi}^{--}\left(p, u^{\pm}\right) + D^{--}\xi^{++}\left(p, u^{\pm}\right), \tag{7.40}$$

where the function $\overline{\xi}^{--}\left(p, u^{\pm}\right)$ is the complex conjugate of $\xi^{++}\left(p, u^{\pm}\right)$. Note also that thinking about $U^{\prime\pm\gamma}$ as $\sqrt{p'}u^{\pm\gamma}$, that is

$$U^{\prime\pm\gamma} = u^{\pm\gamma}\sqrt{F\left(p\right)} \tag{7.41}$$

and $u^{\prime\pm\gamma} = u^{\pm\gamma}$, one sees that the local deformations of the shape of S^3 comes from the variation dp and read as,

$$\begin{aligned} dp \quad &\to \quad dp' \\ dp' \quad &= \quad \left(1 - \frac{\partial\xi}{\partial p}\right)dp + \left(D^{++}\xi\right)d\tau^{--} + \left(D^{--}\xi\right)d\tau^{++} + \left(D^{0}\xi\right)d\tau^{0} \quad , \end{aligned} \tag{7.42}$$

where D^{++}, D^{--} and D^{0} are as in eqs(4.54) and

$$\begin{aligned} d\tau^{++} &= u^{+}du^{+} \quad , \\ d\tau^{--} &= u^{-}du^{-} \quad , \\ d\tau^{0} &= \frac{1}{2}\left(u^{+}du^{-}-u^{-}du^{+}\right) \quad . \end{aligned} \tag{7.43}$$

Like for the conifold analysis, one may use the fibration $S^1 \times S^2$ to rewrite the local deformation parameter $\xi\left(p, u^{\pm}\right)$ as given by the projective function $\varphi\left(p; \sigma, u^{+}, u^{-}\right)$ with $\sigma = e^{i\theta}$ and $u^{+\gamma}u_{\gamma}^{-} = 1$ together with the identification

$$u^{+\gamma} \equiv e^{i\phi}u^{+\gamma}, \qquad u_{\gamma}^{-} \equiv e^{-i\phi}u_{\gamma}^{-}, \qquad \theta \equiv \theta - \phi \quad , \tag{7.44}$$

and

$$\begin{aligned} \varphi\left(p;\sigma,u^{+},u^{-}\right) &= \varphi\left(p;e^{-i\phi}\sigma,e^{i\phi}u^{+},e^{-i\phi}u^{-}\right) \quad , \\ \frac{\partial}{i\partial\theta} &\equiv D^{0}, \qquad d\theta \equiv d\tau^{0} \quad . \end{aligned} \tag{7.45}$$

In the projective coordinates $\left(p; \sigma, u^{+}, u^{-}\right)$, the projective function $\varphi\left(p; \sigma, u^{+}, u^{-}\right)$ may be expanded twice: First in a Fourrier series as follows,

$$\begin{aligned} \varphi\left(p;\sigma,u^{+},u^{-}\right) &= \phi^{0}\left(p;u^{+},u^{-}\right) + \sum_{n>0} e^{in\theta}\ \phi^{+n}\left(p;u^{+},u^{-}\right) \\ &\quad + \sum_{n>0} e^{-in\theta}\ \phi^{-n}\left(p;u^{+},u^{-}\right) \quad , \end{aligned} \tag{7.46}$$

with

$$D^{0}\ \phi^{\pm n}\left(p;u^{+},u^{-}\right) = \pm n\ \phi^{\pm n}\left(p;u^{+},u^{-}\right) \quad . \tag{7.47}$$

Second in a harmonic series on the 2-sphere. In perturbation approach, the projective function $\varphi\left(p; \sigma, u^{+}, u^{-}\right)$ and its Laurent modes $\phi^{\pm n}\left(p; u^{+}, u^{-}\right)$ are treated as

$$\begin{aligned} \varphi\left(p;\sigma,u^{\pm}\right) &= \phi\left(p;\sigma,u^{+}\right) + \overline{\phi}\left(p;\sigma,u^{-}\right) + \mathcal{O}\left(\phi\overline{\phi}\right) \quad , \\ \phi^{+n}\left(p;u^{\pm}\right) &= p^{\frac{n}{2}}u^{(+n)}\zeta_{n} + \delta\eta^{+n}\left(p;u^{\pm}\right) \quad , \\ \phi^{-n}\left(p;u^{\pm}\right) &= p^{\frac{n}{2}}u^{(-n)}\overline{\zeta}_{-n} + \delta\eta^{-n}\left(p;u^{\pm}\right) \quad , \\ \delta\eta^{\pm n}\left(p;u^{\pm}\right) &= \delta\eta^{\pm n}\left(\zeta,\overline{\zeta}\right) \quad , \end{aligned} \tag{7.48}$$

where we have set,

$$\begin{aligned} u^{(+n)} &\equiv u^{+(\alpha_1}...u^{+\alpha_n)}, \qquad u^{(-n)} \equiv u_{(\beta_1}^{-}...u_{\beta_n)}^{-} \quad , \\ \zeta_{n} &\equiv \zeta_{(n,0)} \equiv \zeta_{(\alpha_1...\alpha_n)}, \qquad \overline{\zeta}_{-n} \equiv \zeta^{(\beta_1...\beta_n)} \equiv \zeta_{(0,n)} \quad . \end{aligned} \tag{7.49}$$

Furthermore, referring to S^3 cosmology analysis of previous subsection, one can write down the the harmonic volume 3-form Ω of the deformed 3-sphere. Using the standard relation

$$\Omega = \frac{\Omega_0}{\partial F/\partial p}, \tag{7.50}$$

we get

$$\Omega = \frac{\Omega_0}{1 - \left(\frac{\partial \varphi}{\partial p}\right)} \quad . \tag{7.51}$$

Comparing with $\exp(\Phi) = \Omega/\Omega_0$, we obtain the relation between the conformal factor $\Phi(p, u^{\pm})$ and the local complex deformation function $\varphi(p, u^{\pm})$,

$$\Phi\left(p, u^{\pm}\right) = -\ln\left(1 - \frac{\partial \varphi}{\partial p}\right) = -\ln\left(1 - \frac{\partial \phi}{\partial p} - \frac{\partial \overline{\phi}}{\partial p} - \mathcal{O}\left(\phi\overline{\phi}\right)\right) \quad . \tag{7.52}$$

Upon splitting $\Phi(p, u^{\pm})$ as the sum over the field variable $\Phi(p, u^{+})$ and its complex conjugate $\overline{\Phi}(p, u^{-})$, we get at the first order in ζ,

$$\Phi\left(p, u^{+}\right) \simeq \frac{\partial \phi}{\partial p}, \qquad \overline{\Phi}\left(p, u^{-}\right) \simeq \frac{\partial \overline{\phi}}{\partial p} \quad . \tag{7.53}$$

For later use, it is interesting to note the two following useful relations. First, observe that the quadratic functional field quantity,

$$\frac{1}{g_s^2} S_2 = \frac{2}{g_s^2} \sum_{n>0} \frac{p^n}{n} \left(\int_{S^2} \overline{\zeta}^{-n}\left(u^{-}\right) \; \zeta^{+n}\left(u^{+}\right)\right) \quad , \tag{7.54}$$

may be usually expressed as a harmonic integral over the real 3-sphere. One way to do it is to use the relation

$$p \frac{\partial}{\partial p}\left(p^{\frac{n}{2}} \zeta^{+n}\left(u^{+}\right)\right) = \frac{n}{2}\left(p^{\frac{n}{2}} \zeta^{+n}\left(u^{+}\right)\right) \tag{7.55}$$

to put above integral into the form

$$\frac{1}{g_s^2} \int_{S^3} \overline{\zeta}\left(p, u^{-}\right) \frac{1}{p\frac{\partial}{\partial p}} \zeta\left(p, u^{+}\right). \tag{7.56}$$

Note that we have the operator $p\partial/\partial p$ counting the number of p's reads in terms of the harmonic coordinates as follows,

$$p\frac{\partial}{\partial p} = D^{++}D^{--} + D^{--}D^{++} \equiv \left\{D^{++}, D^{--}\right\}. \tag{7.57}$$

Using the obvious identities,

$$D^{++}\zeta\left(p, u^{+}\right) = 0 \quad , \qquad D^{--}\overline{\zeta}\left(p, u^{-}\right) = 0, \tag{7.58}$$

expressing respectively holomorphy in u^{+} and u^{-}, and standard relations on pseudo-differential operator analysis, in particular

$$\partial^{-k} f = \sum_{s \geq 0} a_s \partial^s f \partial^{-s-k} \tag{7.59}$$

with somes numbers a_s, one can rewrite eq(7.56) as,

$$\frac{1}{g_s^2}\left(\int_{S^3}\zeta\left(p,u^+\right)\frac{1}{D^{--}D^{++}}\overline{\zeta}\left(p,u^-\right)\right) \quad . \tag{7.60}$$

Then by integration by part, we can bring it to,

$$\frac{1}{g_s^2}S_2=\frac{-1}{g_s^2}\left[\int_{S^3}\left(\frac{1}{D^{--}}\zeta\left(p,u^+\right)\right)\left(\frac{1}{D^{++}}\overline{\zeta}\left(p,u^-\right)\right)\right] \quad . \tag{7.61}$$

In this relation the measure $\int_{S^3}$ refers to the normalized integral over the three sphere

$$\int_{S^3}1=1 \tag{7.62}$$

which factorises in terms of normalized measures on S^1 and S^2 as

$$\int_{S^2}\left(\int_{S^1}1\right)=1. \tag{7.63}$$

Similarly, we can express the dimensionless cubic functional field quantity,

$$\frac{1}{g_s^2}S_3=-\frac{1}{6g_s^2}\sum_{n_1+n_2=n_3}p^{\frac{|n_1|+|n_2|+|n_3|}{2}-1}\left(\int_{S^2}\zeta^{+n_1}\zeta^{+n_2}\overline{\zeta}^{-n_3}+\overline{\zeta}^{-n_1}\overline{\zeta}^{-n_2}\zeta^{+n_3}\right) \quad , \tag{7.64}$$

like a harmonic integral over the three sphere as shown below,

$$\frac{1}{g_s^2}S_3=\frac{1}{6g_s^2}\left(\frac{1}{p}\int_{S^3}\zeta^2\left(p,u^+\right)\overline{\zeta}\left(p,u^-\right)+\zeta\left(p,u^+\right)\overline{\zeta}^2\left(p,u^-\right)\right) \quad . \tag{7.65}$$

Note that these S_2 and S_3 terms, which will be used later, are in fact just the leading two terms of a series

$$S=\sum_{n\geq 2}S_n \tag{7.66}$$

describing the perturbative expansion of genus zero free energy $\mathcal{F}_0$ of the partition function of the topological string B model on conifold. To prove the statements (7.61,7.65), one needs the harmonic expansions (5.44) and use the following result on the harmonic integrals

$$I_{k,l}\left(x,\zeta,\overline{\zeta}\right)=\int_{S^3}\left(\zeta\right)^k\left(\overline{\zeta}\right)^l \tag{7.67}$$

with k and l two positive integers,

$$I_{k,l}\left(x,\zeta,\overline{\zeta}\right)=\sum_{n_1,\ldots,m_l\neq 0}x^{\frac{|n_1|+\ldots+|n_k|+|m_1|+\ldots+|m_l|}{2}}\delta_{n_1+\ldots+n_k,m_1\ldots m_l}\int_{S^2}\left(\prod_{j=1}^{k}\zeta^{+n_j}\prod_{s=1}^{l}\overline{\zeta}^{-m_s}\right). \tag{7.68}$$

For $k=l=0$, the integral $I_{0,0}\left(x,\zeta,\overline{\zeta}\right)$ is just the normalized volume of S^3 and for $k=1$, $l=0$, the integral $I_{1,0}\left(x,\zeta,\overline{\zeta}\right)$ vanishes identically as there is no $SU\left(2,C\right)$ singlet within

ζ. For $(k,l) = (1,1)$ and $(2,1)$, we have the following

$$I_{1,1}\left(x,\zeta,\overline{\zeta}\right) = \sum_{n\neq 0} x^{|n|}\left(\int_{S^2}\zeta^{+n}\overline{\zeta}^{-n}\right), \tag{7.69}$$

$$I_{2,1}\left(x,\zeta,\overline{\zeta}\right) = \sum_{n_1+n_2\neq 0} x^{\frac{|n_1|+|n_2|+|n_1+n_2|}{2}}\left(\int_{S^2}\zeta^{+n_1}\zeta^{+n_2}\overline{\zeta}^{-n_1-n_2}\right). \tag{7.70}$$

Putting altogether these relations, one gets the desired results.

7.3 Correlation functions

In the harmonic frame work of the GSV model of S^3 quantum cosmology, we can compute the $SU(2,C)$ manifestly covariant N points Green functions $G_N = G\left(U_1^{\pm},...,U_N^{\pm}\right)$,

$$G_N =< \Phi\left(U_1^{\pm}\right)...\Phi\left(U_N^{\pm}\right) > \quad . \tag{7.71}$$

These functions describe the correlations between the conformal fields $\Phi_i = \Phi\left(U_i^{\pm}\right)$ at the points $U_i^{\pm}$ with

$$U_i^{+\alpha}U_{\alpha i}^{-} = p \tag{7.72}$$

and their evaluation uses Hartle-Hawking probability density $\varrho\left(p,\zeta,\overline{\zeta}\right) = \left|\Psi\left(p,\zeta,\overline{\zeta}\right)\right|^2$.

7.3.1 Hartlee-Hawking probability density

The probability density $\varrho\left(p,\zeta,\overline{\zeta}\right)$ involves the Hartle-Hawking universe wave function

$$\Psi = \Psi\left(p,\zeta,\overline{\zeta}\right) \tag{7.73}$$

whose expression has been shown to coincide with the topological string partition function

$$\left|\mathcal{Z}_{top}\left(p,\zeta,\overline{\zeta}\right)\right|^2 = \exp 2\,\mathrm{Re}\left(\mathcal{F}\left(p,\zeta,\overline{\zeta}\right)\right) \quad . \tag{7.74}$$

With this result in mind and following [30] as well as using the analysis of previous section and standard techniques from quantum field theory, the generic N points Green function G_N (7.71) may be also formulated as follows,

$$G_N = \mathcal{N}\int D\zeta D\overline{\zeta}\left(\prod_{i=1}^{N}\Phi_i\left(U_i^{\pm}\right)\right)\exp\left(-\frac{1}{g_s^2}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]\right) \quad , \tag{7.75}$$

where

$$D\zeta D\overline{\zeta} = \prod_{n\geq 0}\left(d\zeta^{+n}d\overline{\zeta}^{-n}\right) \quad , \tag{7.76}$$

the hermitian field Φ_i standing for the infinitesimal complex field variable $\Phi\left(p_i,u_i^{+}\right)$ together with its complex conjugate $\overline{\Phi}\left(p_i,u_i^{-}\right)$ and where we have set,

$$\varrho\left(p,\zeta,\overline{\zeta}\right) = \mathcal{N}\exp\left(-\frac{1}{g_s^2}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]\right). \tag{7.77}$$

The normalization factor $\mathcal{N}$ is fixed as usual by the unitary condition

$$\mathcal{N}\int D\zeta D\overline{\zeta}\exp\left(-\frac{1}{g_s^2}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]\right)=1 \tag{7.78}$$

and, in genus zero approximation of free energy, the factor $\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]$ is defined as,

$$\begin{aligned}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right] &= \sum_{n>0}\frac{2}{n}p^n\left(\int_{S^2}\zeta^{+n}\overline{\zeta}^{-n}\right)\\ &\quad -\frac{1}{3}\sum_{n_1+n_2=n_3}p^{\frac{|n_1|+|n_2|+|n_3|-2}{2}}\left(\int_{S^2}\zeta^{+n_1}\zeta^{+n_2}\overline{\zeta}^{-n_3}\right)+O\left(\left[\zeta^4\right]\right).\end{aligned} \tag{7.79}$$

Observe that under the change

$$p\longrightarrow\lambda p,\qquad \zeta\longrightarrow\lambda\zeta,\qquad g_s\longrightarrow\lambda g_s,\qquad \zeta^{\pm n}\longrightarrow\lambda^{1-\frac{|n|}{2}}\zeta^{\pm n}, \tag{7.80}$$

$\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]$ scales as,

$$\mathcal{S}\left[\lambda p,\lambda\zeta,\lambda\overline{\zeta}\right]=\lambda^2\mathcal{S}\left[p,\zeta,\overline{\zeta}\right], \tag{7.81}$$

and so the ratio $\frac{1}{g_s^2}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]$ appearing in the exponential of the Hartlee-Hawking probability density is invariant.

7.3.2 $\frac{1}{p}$ expansion

Using the identities (7.54-7.65), eq(7.66) may be also formulated as follows,

$$\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]=\int_{S^3}\zeta\frac{1}{\{D^{--},D^{++}\}}\overline{\zeta}-\frac{1}{6p}\int_{S^3}\left(\zeta^2\overline{\zeta}+\zeta\overline{\zeta}^2\right)+\mathcal{O}\left(\frac{1}{p^2}\left(\zeta\overline{\zeta}\right)^2,...\right)\quad. \tag{7.82}$$

This expression recalls the usual field theory action; but with a non local operator for the quadratic ζ term. It may be imagined as a field theory action with a tower interacting terms ζ^n and coupling constant λ_n proportional to

$$\frac{1}{p^{n-2}}.$$

In the limit of a 3-sphere with large volume, one may treat these interacting terms as perturbations around the Gaussian like factor,

$$\mathcal{S}_0\left[p,\zeta,\overline{\zeta}\right]=\int_{S^3}\zeta\frac{1}{\{D^{--},D^{++}\}}\overline{\zeta}, \tag{7.83}$$

which we discuss below.

Thinking about the normalization condition of the probability density $\rho\left[p,\zeta,\overline{\zeta}\right]$ as just the value $\mathcal{Z}\left[p,J=0\right]$ of some partition function $\mathcal{Z}\left[p,\mathrm{J}\right]$ defined as,

$$\mathcal{Z}\left[p,\mathrm{J},\overline{\mathrm{J}}\right]=\mathcal{N}\int D\zeta D\overline{\zeta}\exp\left(-\frac{1}{g_s^2}\mathcal{S}\left[p,\zeta,\overline{\zeta}\right]+\int_{S^3}\left(\mathrm{J}\zeta+\overline{\mathrm{J}\zeta}\right)\right), \tag{7.84}$$

where J and its complex conjugate $\overline{\mathrm{J}}$ are local external source fields living on the 3-sphere, one can compute Green functions type

$$< \zeta\left(p_{1}, u_{1}\right) ... \zeta\left(p_{N_{1}}, u_{N_{1}}\right) \overline{\zeta}\left(p_{N_{1}+1}, u_{N_{1}+1}\right) ... \overline{\zeta}\left(p_{N_{1}+N_{2}}, u_{N_{1}+N_{2}}\right) > \tag{7.85}$$

and more generally

$$< \left(\prod_{i=1}^{N_{1}} \frac{\partial^{n_{i}} \zeta\left(p_{i}, u_{i}^{\pm}\right)}{\partial p_{i}^{n_{i}}}\right)\left(\prod_{j=1}^{N_{2}} \frac{\partial^{m_{j}} \zeta\left(p_{j}, u_{j}^{\pm}\right)}{\partial p_{j}^{m_{j}}}\right) >, \qquad p_{i}=p_{j}=p, \tag{7.86}$$

describing the correlations between the field variable $\zeta(x, u)$, $\overline{\zeta}(x, u)$ and their derivatives. For $< \zeta\left(p_{1}, u_{1}^{+}\right) ... \overline{\zeta}\left(p_{N}, u_{N}^{-}\right) >$, we have,

$$< \zeta\left(p_{1}, u_{1}^{+}\right) ... \overline{\zeta}\left(p_{N}, u_{N}^{-}\right) >= \frac{\delta^{N} \mathcal{Z}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]}{\delta \mathrm{J}\left(p_{1}, u_{1}^{+}\right) ... \delta \overline{\mathrm{J}}\left(p_{N}, u_{N}^{-}\right)}|_{\mathrm{J}=\overline{\mathrm{J}}=0}. \tag{7.87}$$

To compute these correlation functions, it is interesting to bring the above generating functional $\mathcal{Z}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]$ into a manageable form. With the idea of a perturbation theory in $(1/p)$ in mind and using the usual trick

$$\zeta\left(p, u^{\pm}\right)=\frac{\delta}{\delta \mathrm{J}\left(p, u^{\pm}\right)}, \tag{7.88}$$

the above relation can be also expressed as,

$$\mathcal{Z}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]=\mathcal{N}\left[\exp \left(\frac{1}{6 p} \int_{S^{3}}\left(\frac{\delta^{3}}{\delta \mathrm{J}^{2}\left(p, u^{+}\right) \delta \overline{\mathrm{J}}\left(p, u^{-}\right)}+cc\right)+O(4)\right)\right] \mathcal{Z}_{0}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right], \tag{7.89}$$

where,

$$\mathcal{Z}_{0}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]=\mathcal{N} \int D \zeta D \overline{\zeta} \exp \left(-\frac{1}{g_{s}^{2}} \mathcal{S}_{0}\left[p, \zeta, \overline{\zeta}\right]+\int_{S^{3}}(\mathrm{J} \zeta+cc)\right). \tag{7.90}$$

Up on integrating with respect to the field variable ζ, $\mathcal{Z}_{0}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]$ reads also as,

$$\mathcal{Z}_{0}\left[p, \mathrm{J}, \overline{\mathrm{J}}\right]=\exp \left[-\frac{g_{s}^{2}}{2} \int_{S_{i}^{3} \times S_{j}^{3}} \overline{\mathrm{J}}\left(p_{i}, u_{i}^{-}\right)\left[\delta\left(u_{i}^{\pm}, u_{j}^{\pm}\right)\left\{D_{j}^{++}, D_{j}^{--}\right\}\right] \mathrm{J}\left(p_{i}, u_{j}^{+}\right)\right], \tag{7.91}$$

where $\delta\left(u_{i}^{\pm}, u_{j}^{\pm}\right)$ is a harmonic distribution defined as,

$$\int_{S_{j}^{3}} \delta\left(u_{i}^{\pm}, u_{j}^{\pm}\right) F\left(p_{j}, u_{j}^{\pm}\right)=F\left(p_{i}, u_{i}^{\pm}\right). \tag{7.92}$$

The free field propagators

$$< \zeta\left(p_{i}, u_{i}^{+}\right) \overline{\zeta}\left(p_{j}, u_{j}^{-}\right) >_{0} \tag{7.93}$$

and the corresponding correlation function

$$< \Phi\left(p_{i}, u_{i}^{\pm}\right) \overline{\Phi}\left(p_{j}, u_{j}^{\mp}\right) >_{0} \tag{7.94}$$

as well as higher order points Green functions are easily obtained from eq(7.91). For example, we have

$$< \zeta \left(p_i, u_i^+\right) \zeta \left(p_j, u_j^+\right) >= 0. \tag{7.95}$$

The same result is valid for any correlation function involving one handed complex deformations as shown below,

$$\begin{aligned} &< \prod_{i=1}^{N_1} \frac{\partial^{n_i} \zeta \left(p_i, u_i^{\pm}\right)}{\partial p_i^{n_i}} >= 0, \\ &< \prod_{j=1}^{N_2} \frac{\partial^{m_j} \zeta \left(p_j, u_j^{\pm}\right)}{\partial p_j^{m_j}} >= 0. \end{aligned} \tag{7.96}$$

The simplest non trivial results are given by the propagator,

$$< \zeta \left(p_i, u_i^+\right) \overline{\zeta} \left(p_j, u_j^-\right) >_0 = g_s^2 \left\{D_j^{++}, D_j^{--}\right\} \delta \left(u_i^{\pm}, u_j^{\pm}\right), \tag{7.97}$$

We also have,

$$< \Phi \left(p_i, u_i^+\right) \overline{\Phi} \left(p_j, u_j^-\right) >_0 = \frac{g_s^2}{p^2} \left(\left\{D_j^{++}, D_j^{--}\right\}\right)^3 \delta \left(u_i^{\pm}, u_j^{\pm}\right). \tag{7.98}$$

Using these relations, one can compute all correlation functions. All of them preserve manifestly the $SU(2, C)$ isometry of the 3-sphere.

8 Conclusion

In this paper we have studied the partition function $\mathcal{Z}_{top}$ of the B-model topological string on conifold and the correlations functions of the scale operator field of Gukov-Saraikin-Vafa quantum cosmology model on S^3. To that purpose, we have first developed harmonic analysis for conifold and shown that this formalism gives in fact a unified description of T^*S^3 and its sub-varieties T^*S^2, S^3 and S^2. Recall that the idea of using harmonic space has been considered in the past by Galperin *et al* for the case of S^2 for the construction of an off shell superfield formulation of extended supersymmetic gauge theories with $SU_R(2)$ symmetry. The harmonic space method developed in this paper goes beyond and has the following basic properties:

(**1**) It preserves manifestly conifold $SL(2, C)$ isometry and allows to complete partial results for the computation of the partition function of topological strings on conifold. Recall that in standard complex analysis, computation of $\mathcal{Z}_{top}(T^*S^3)$ deals with the subset of conifold local complex deformations restricted to its subspace T^*S^1. Harmonic space method for conifold covers local complex deformation over all conifold points. Moreover the harmonic variables U_α^+ and V_β^- get a remarkable interpretation in 2D

$c = 1$ non critical string theory. There U_1^+ and U_2^+ are respectively associated with positive unit momentum and winding modes. V_1^- and V_2^- are associated with negative unit mode momentum and winding.

(**2**) The harmonic method applies as well for the study of restrictions down to $T^*S^2 \sim SL(2)/C^*$ and to the real slices $S^3 \sim SU(2)$ and $S^2 \sim SU(2)/U(1)$. The harmonic formalism for T^*S^2 is recovered naturally from that of T^*S^3 by fixing the C^* symmetry of T^*S^1 fiber. This harmonic space method may be applied as well for studying local complex deformation moduli space of string on $K3$ with deformed A_1 singularity.

(**3**) Harmonic space analysis on T^*S^3 has a remarkable 1 to 1 correspondence Laurent analysis on T^*S^1. This property, which is also valid for harmonic analysis S^3 and Fourier analysis on S^1, allowed us to :

(**a**) construct the dictionary (5.59) giving the correspondence between T^*S^3 and T^*S^1 and similarly between their real compact slices.

(**b**) use the dictionary (5.59) to derive the explicit form of the B-model topological string partition function on conifold. This obtained partition function preserves manifestly conifold $SL(2)$ isometry.

With this machinery at hand, and guided by known results on the ground ring of 2D $c = 1$ string theory [16], we have reconsidered the study the local complex deformations of T^*S^3 keeping track with manifest $SL(2,C)$ isometry of the conifold. We have also derived the explicit expression of the manifestly $SL(2,C)$ invariant form of the partition function $\mathcal{Z}_{top}$ of B-model topological string on conifold; see also theorem of section 7. The harmonic space method has been also used to study stringy quantum cosmology of Gukov Saraikin and Vafa; in particular to compute the manifestly $SU(2,C)$ covariant correlation functions $< \Phi(S^3) ... \Phi(S^3) >$ of the GSV quantum cosmology model. Actually this analysis completes the results of [30]

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9 Appendix: Harmonic analysis on conifold and subvarieties

In this section, we give some useful details on the harmonic analysis for conifold T^*S^3, the cotangeant bundle of complex one dimension projective space T^*P^1 and their corresponding real slices namely the spheres S^3 and S^2. We give the main lines on harmonic expansions on these manifolds, the harmonic differential calculus, forms, harmonic integration and the various types of harmonic distributions.

9.1 General set up

We begin by recalling that in harmonic frame work, the conifold $T^{*}S^{3}$ is defined by

$$u^{+\alpha}v_{\alpha}^{-}=1, \qquad u^{+\alpha}u_{\alpha}^{+}=0, \qquad v^{-\alpha}v_{\alpha}^{-}=0 \tag{9.1}$$

with $\left(u^{+1},u^{+2},v_{1}^{-},v_{2}^{-}\right)$ four complex coordinates in C^{4} and where, for simplicity, we have set $\mu=1$. Its two complex dimension coset

$$T^{*}P^{1}\sim T^{*}S^{3}/C^{*} \tag{9.2}$$

is defined by similar relations,

$$u^{+\alpha}v_{\alpha}^{-}=1, \qquad u^{+\alpha}u_{\alpha}^{+}=0, \qquad v^{-\alpha}v_{\alpha}^{-}=0, \tag{9.3}$$

but now $u^{+\alpha}$ and v_{α}^{-} belong to to the weighted projective space $WP^{3}\left(1,1,-1,-1\right)$, i.e with projective transformations,

$$u^{+\alpha}\rightarrow\lambda u^{+\alpha}, \qquad v_{\alpha}^{-}\rightarrow\frac{1}{\lambda}v_{\alpha}^{-}, \qquad \lambda\in C^{*}. \tag{9.4}$$

For later use, note that the variable v_{α}^{-} in the global defined equation $u^{+\alpha}v_{\alpha}^{-}=1$ may be locally solved in the vicinity of $u^{+\alpha}=\eta^{+\alpha}$ and $v_{\alpha}^{-}=\eta_{\alpha}^{-}$ as follows

$$v_{\alpha}^{-}=\frac{1}{u^{+\alpha}\eta_{\alpha}^{+}}\eta_{\alpha}^{+}, \qquad \eta^{+\alpha}\eta_{\alpha}^{-}=1 \qquad \eta^{\pm\alpha}\eta_{\alpha}^{\pm}=0. \tag{9.5}$$

Harmonic functions $\mathcal{F}$ on conifold are defined by help of harmonic functions $F^{q}=F^{q}(u^{+},v^{-})$ living on $T^{*}P^{1}$. These are homogeneous holomorphic functions

$$F^{q}(\lambda u^{+},\frac{1}{\lambda}v^{-})=\lambda^{q}F^{q}(u^{+},v^{-}) \tag{9.6}$$

depending on the harmonic variables u^{+} and v^{-}; but no dependence on u^{-} and v^{+}; i.e,

$$\frac{\partial F^{q}}{\partial u^{-}}=0, \qquad \frac{\partial F^{q}}{\partial v^{+}}=0. \tag{9.7}$$

To get the expansion of $\mathcal{F}$, one thinks about conifold as embedded in the projective space $WP^{4}\left(-1,1,1,-1,-1\right)$ with the projective coordinates

$$\sigma\equiv\frac{1}{\lambda}\sigma, \qquad u^{+\alpha}\equiv\lambda u^{+\alpha}, \qquad v_{\alpha}^{-}\equiv\frac{1}{\lambda}v_{\alpha}^{-}. \tag{9.8}$$

with λ a C^{*} gauge parameter. Function $\mathcal{F}\left(\sigma,u^{+},v^{-}\right)$ on conifold are projective function

$$\mathcal{F}\left(\frac{1}{\lambda}\sigma,\lambda u^{+},\frac{1}{\lambda}v^{-}\right)=\mathcal{F}\left(\sigma,u^{+},v^{-}\right) \tag{9.9}$$

and have then the following Laurent expansion,

$$F\left(\sigma, u^{+}, v^{-}\right)=\sum_{q \in Z} \sigma^{q} F^{q}(u^{+}, v^{-}) \tag{9.10}$$

where $F^{q}(u^{+}, v^{-})$ are as before. Laurent modes

$$F^{q}(u^{+}, v^{-})=\oint \frac{d\sigma}{2i\pi} \sigma^{-q-1} \mathcal{F}\left(\sigma, u^{+}, v^{-}\right) \tag{9.11}$$

carry a well defined C^{*} integer charge q as shown below,

$$\nabla^{0} F^{q}=\left(u^{+\alpha} \frac{\partial}{\partial u^{+\alpha}}-v^{-\alpha} \frac{\partial}{\partial v^{-\alpha}}\right) F^{q}=q F^{q}, \tag{9.12}$$

and have the following typical harmonic expansion

$$F^{q}(u^{+}, v^{-})=\sum_{n \geq 0} F^{(\alpha_{1} \ldots \alpha_{q+n} \beta_{1} \ldots \beta_{n})}\ u_{(\alpha_{1}}^{+} \ldots u_{\alpha_{q+n}}^{+} v_{\beta_{1}}^{-} \ldots v_{\beta_{n})}^{-}. \tag{9.13}$$

In this relation, the harmonic monomial quantity

$$u_{(\alpha_{1}}^{+} \ldots u_{\alpha_{n}}^{+} v_{\beta_{1}}^{-} \ldots v_{\beta_{m})}^{-} \tag{9.14}$$

stands for a complete symmetrization of the harmonic variables,

$$u_{(\alpha_{1}}^{+} \ldots u_{\alpha_{n}}^{+} v_{\beta_{1}}^{-} \ldots v_{\beta_{m})}^{-}=\frac{1}{(n+m)!} \sum_{\sigma \in \mathcal{S}_{n+m}} u_{(\alpha_{\sigma(1)}}^{+} \ldots u_{\alpha_{\sigma(n)}}^{+} v_{\beta_{\sigma(1)}}^{-} \ldots v_{\beta_{\sigma(m)})}^{-}, \tag{9.15}$$

and $F^{(\alpha_{1} \ldots \alpha_{q+n} \beta_{1} \ldots \beta_{n})}$ stands for the harmonic modes obtained by inverting the relation. This is achieved by help of harmonic integration rules on $T^{*}P^{1}$ which we describe below. Before, note that the identification between the operator ∇^{0} and $\sigma \frac{\partial}{\partial \sigma}$ follows from the splitting $SL\left(2\right)=C^{*} \times SL\left(2\right)/C^{*}$.

9.1.1 Harmonic differential forms

Differential forms on the conifold, in particular harmonic 1-forms, preserving manifestly the $SL\left(2, C\right)$ isometry group covariance may be also obtained by starting from the holomorphic 1-forms

$$du^{+\beta} \tag{9.16}$$

and the anti-holomorphic ones

$$dv_{\beta}^{-} \tag{9.17}$$

on the ambient complex space C^{4} and build the appropriate isotriplet of 1-forms where $SL\left(2, C\right)$ indices are contracted. One way to do it is to note that instead of $du^{+\beta}$ and dv_{β}^{-} one may use the equivalent expressions

$$u^{+\alpha} du_{\alpha}^{+}, \qquad v_{\alpha}^{-} dv^{-\alpha}, \qquad \frac{1}{2}\left(v_{\alpha}^{-} du^{+\alpha}-v_{\alpha}^{-} du^{+\alpha}\right), \tag{9.18}$$

$$\frac{1}{2}\left(u^{+\alpha} dv_{\alpha}^{-}+v_{\alpha}^{-} du^{+\alpha}\right). \tag{9.19}$$

The three first 1-forms constitute altogether an $SL(2,C)$ isotriplet; while the fourth one, which reads also as

$$\frac{1}{2}d\left(u^{+\alpha}v_{\alpha}^{-}\right), \tag{9.20}$$

is a $SL(2,C)$ isosinglet. On the conifold and $T^{*}P^{1}$ where $u^{+\alpha}v_{\alpha}^{-}$ is a constant, the differential $d\left(u^{+\alpha}v_{\alpha}^{-}\right)$ vanishes identically and one ends with the following harmonic one forms,

$$\begin{aligned} d\tau^{(++,0)} &= \varepsilon_{\alpha\beta}u^{+\alpha}du^{+\beta}, \\ d\tau^{(+,-)} &= \frac{1}{2}\left(u^{+\alpha}dv_{\alpha}^{-}-v_{\alpha}^{-}du^{+\alpha}\right), \\ d\tau^{(0,--)} &= \varepsilon^{\alpha\beta}v_{\alpha}^{-}dv_{\beta}^{-}, \end{aligned} \tag{9.21}$$

where the convention notation (p,q) refers to p charges u^{+} and q charges v^{-}. Note that because of the constraint eq $u^{+\alpha}v_{\alpha}^{-}=1$, the 1-form $d\tau^{(+,-)}$ reads also as $u^{+\alpha}dv_{\alpha}^{-}$ or equivalently $v_{\alpha}^{-}du^{+\alpha}$.

In the coordinate frame $(\sigma,u^{+\alpha},v_{\alpha}^{-})$ of $WP^{4}(-1,1,1,-1,-1)$, the one forms $d\tau^{(++,0)}$ and $d\tau^{(0,--)}$ map respectively as $\lambda^{2}d\tau^{(++,0)}$ and $\lambda^{-2}d\tau^{(0,--)}$ and the non covariant $d\tau^{(+,-)}$ gets identified locally with

$$\frac{d\sigma}{\sigma}. \tag{9.22}$$

Using eq(9.5), one may express the local identification as $\sigma=u^{+\alpha}\eta_{\alpha}^{+}$ and so we have,

$$\oint v_{\alpha}^{-}du^{+\alpha}=\oint\frac{d\left(u^{+\alpha}\eta_{\alpha}^{+}\right)}{u^{+\alpha}\eta_{\alpha}^{+}}=\oint\frac{d\sigma}{\sigma}\sim 1. \tag{9.23}$$

In the frame $(\sigma,u^{+\alpha},v_{\alpha}^{-})$ frame the projective one forms $d\tau^{(++,0)}$ and $d\tau^{(0,--)}$ are the holomorphic 1-forms on $T^{*}P^{1}$. The holomorphic volume 2-form of $T^{*}P^{1}$ is

$$d\tau^{(++,0)}\wedge d\tau^{(0,--)} \tag{9.24}$$

and the holomorphic three form Ω on the conifold reads then as

$$\frac{d\sigma}{\sigma}\wedge d\tau^{(0,--)}\wedge d\tau^{(++,0)}. \tag{9.25}$$

The results on the real slices S^{3} and S^{2} are recovered by setting $v^{-}=u^{-}$ and underlying constraints. In what follows, we consider some aspect on this real truncation.

9.1.2 3-sphere

Let us start by recalling that in the harmonic frame work one of the remarkable ways to define the volume form ω_{0} of a *unit* three sphere is as follows,

$$\omega_{0}=A\ d\tau^{0}\wedge d\tau^{--}\wedge d\tau^{++},\qquad \int_{S^{3}}\omega_{0}=1, \tag{9.26}$$

where one recognizes

$$d\tau^{++} \wedge d\tau^{--} \tag{9.27}$$

as the real volume 2-form on the real 2-sphere. The advantage of this definition is that all harmonic variables are treated on equal footing. Moreover ω_0 is manifestly $SU\left(2,C\right)$ invariant since it is gauge invariant under projective transformation

$$u^{\pm\alpha\prime} = e^{\pm i\theta} u^{\pm\alpha}, \tag{9.28}$$

a property captured by the sum of Cartan Weyl charge which add exactly to zero. To have more insight in this way of doing, note that for a unit sphere parameterized by small harmonics

$$u^{+\alpha} u_{\alpha}^{-} = 1, \qquad u^{\pm\alpha} u_{\alpha}^{\pm} = 0, \tag{9.29}$$

the harmonic 1-forms $d\tau^{(q)}$, $q = 0, \pm 2$ read as

$$\begin{aligned} d\tau^{++} &= 2u^{+\alpha} du_{\alpha}^{+} = 2\varepsilon_{\alpha\beta} u^{+\alpha} du^{+\beta}, \\ d\tau^{--} &= 2u^{-\alpha} du_{\alpha}^{-} = 2\varepsilon^{\alpha\beta} u_{\beta}^{-} du_{\alpha}^{-}, \\ d\tau^{0} &= \left(u^{+\alpha} du_{\alpha}^{-} - u_{\alpha}^{-} du^{+\alpha}\right). \end{aligned} \tag{9.30}$$

They form altogether an $SU\left(2,C\right)$ vector as one sees by help of eqs(4.54) and the highest weight state condition

$$\left[D^{0}, d\tau^{++}\right] = 2d\tau^{++}, \qquad \left[D^{++}, d\tau^{++}\right] = 0, \tag{9.31}$$

together with the following relations,

$$\begin{aligned} \left[D^{--}, d\tau^{++}\right] &= d\tau^{0} \\ \left[D^{--}, d\tau^{0}\right] &= d\tau^{--}, \\ \left[D^{--}, d\tau^{--}\right] &= 0. \end{aligned} \tag{9.32}$$

The fourth remaining symmetric combination namely

$$u^{+\alpha} du_{\alpha}^{-} + u_{\alpha}^{-} du^{+\alpha} \tag{9.33}$$

vanishes identically due to the constraint eq $u^{+\alpha} u_{\alpha}^{-} = 1$ and because of the property,

$$\left(u^{+\alpha} du_{\alpha}^{-} + u_{\alpha}^{-} du^{+\alpha}\right) = d\left(u^{+\alpha} u_{\alpha}^{-}\right) = 0 \tag{9.34}$$

With help of the above relations, one can easily check the $SU\left(2,C\right)$ invariance of the three form ω_0,

$$\left[D^{0}, \omega_0\right] = \left[D^{++}, \omega_0\right] = \left[D^{--}, \omega_0\right] = 0. \tag{9.35}$$

To establish eq(9.26), one starts as usual from the real volume form v_4 of the complex space C^2 and implement the constraint relations of the real hypersurface. In harmonic language, this corresponds to taking

$$v_4 = A\ du^{+1} \wedge du^{+2} \wedge du_1^- \wedge du_2^- \tag{9.36}$$

where A is a normalization factor to be fixed later; then implement the constraint eqs $u^{+\alpha}u_\alpha^- = 1$ and $u^{\pm\alpha}u_\alpha^\pm = 0$. These harmonic eqs may be handled in different manners; for instance by singling out a harmonic variable say u_2^- and solves it by help of above constraint eqs as

$$u_2^- + \frac{u^{+1}u_1^+}{u^{-2}} = 0, \qquad u_2^- + \frac{u^{+1}u_1^- - 1}{u^{+2}}. \tag{9.37}$$

This method breaks however the $SU(2,C)$ isometry we have been preserving so far. An other way is to use $SU(2,C)$ invariance of v_4 to rewrite like

$$v_4 = \frac{-A}{4}\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta}du^{+\alpha} \wedge du^{+\beta} \wedge du_\gamma^- \wedge du_\delta^- \tag{9.38}$$

and implement the relation $u^{+\alpha}u_\alpha^- = 1$ to first reduces it to

$$\omega_0 = \frac{-A}{8}d\tau^{--} \wedge du^{+\alpha} \wedge du^{+\beta}\varepsilon_{\alpha\beta} \tag{9.39}$$

and then substitute the expression of the antisymmetric tensor

$$\varepsilon_{\alpha\beta} = \left(u_\alpha^- u_\beta^+ - u_\beta^- u_\alpha^+\right) \tag{9.40}$$

to put $\left(du^{+\alpha} \wedge du^{+\beta}\right)\varepsilon_{\alpha\beta}$ like

$$\frac{1}{2}d\tau^0 \wedge d\tau^{++} \tag{9.41}$$

which should be compared with eq(9.26). The factor is fixed by the condition

$$\int_{S^3} \omega_0 = 1. \tag{9.42}$$

Note that under the change $u^\pm \to e^{\pm i\theta}u^\pm$, the harmonic differentials $d\tau^{++}$ and $d\tau^{--}$ transform covariantly as

$$\begin{aligned} d\tau^{++} &\to e^{2i\theta}d\tau^{++}, \\ d\tau^{--} &\to e^{-2i\theta}d\tau^{--} \end{aligned} \tag{9.43}$$

while the non covariant $d\tau^0$ gets identified with one form on circle S^1. The above description extends naturally to the case of real 3-spheres $U^{+\gamma}U_\gamma^- = p$ with generic radii.

9.2 Harmonic integration and distributions

9.2.1 Integration rules

Because of the constraint eqs $u^{+\alpha}v_{\alpha}^{-} = 1$ and $u^{+\alpha}u_{\alpha}^{+} = 0$, $v^{-\alpha}v_{\alpha}^{-}$, one can reduces monomials in harmonic variables, belonging to tensor product representations of isospinors, as a sum of terms transforming in $SL(2,C)$ irreducible representations. These reductions are useful in harmonic integration analysis involving traces on irreducible representations of $SL(2,C)$. There are some typical relations which are particularly interesting in performing integration calculus. A set of these relations corresponds those given by the two following standard reduction formulas,

$$u_{\alpha}^{+}u_{(\beta_1}^{+}..u_{\beta_n}^{+}v_{\gamma_1}^{-}..v_{\gamma_m)}^{-} = u_{(\alpha}^{+}u_{\beta_1}^{+}..v_{\gamma_m)}^{-} + \frac{m}{m+n+1}\varepsilon_{\alpha(\gamma_1}u_{\beta_1}^{+}..u_{\beta_n}^{+}v_{\gamma_2}^{-}..v_{\gamma_m)}^{-}, \tag{9.44}$$

and,

$$v_{\gamma}^{-}u_{(\beta_1}^{+}..u_{\beta_n}^{+}v_{\gamma_1}^{-}..v_{\gamma_m)}^{-} = v_{(\gamma}^{-}u_{\beta_1}^{+}..v_{\gamma_m)}^{-} - \frac{n}{m+n+1}\varepsilon_{\gamma(\beta_1}u_{\beta_2}^{+}..u_{\beta_n}^{+}v_{\gamma_1}^{-}..v_{\gamma_m)}^{-}. \tag{9.45}$$

As an illustration, we give the following leading examples for $n, m = 0, 1$. For $n = 1$ and $m = 0$, we have

$$u_{\alpha}^{+}u_{\beta}^{+} = u_{(\alpha}^{+}u_{\beta)}^{+} \tag{9.46}$$

and

$$v_{\alpha}^{-}u_{\beta}^{+} = v_{(\alpha}^{-}u_{\beta)}^{+} - \frac{1}{2}\varepsilon_{\alpha\beta} \tag{9.47}$$

while for $n = 0$ and $m = 1$, we have

$$v_{\alpha}^{-}v_{\beta}^{-} = v_{(\alpha}^{-}v_{\beta)}^{-} \tag{9.48}$$

and

$$u_{\alpha}^{+}v_{\beta}^{-} = u_{(\alpha}^{+}v_{\beta)}^{-} + \frac{1}{2}\varepsilon_{\alpha\beta}. \tag{9.49}$$

For $n = m = 1$, we have the following reduction,

$$\begin{aligned} u_{\alpha}^{+}u_{(\beta}^{+}v_{\gamma)}^{-} &= u_{(\alpha}^{+}u_{\beta}^{+}v_{\gamma)}^{-} + \frac{1}{3}\varepsilon_{\alpha(\gamma}u_{\beta)}^{+}, \\ v_{\alpha}^{-}u_{(\beta}^{+}v_{\gamma)}^{-} &= v_{(\alpha}^{-}u_{\beta}^{+}v_{\gamma)}^{-} - \frac{1}{3}\varepsilon_{\alpha(\beta}v_{\gamma)}^{-}. \end{aligned} \tag{9.50}$$

These relations are important for integration on harmonic variables. Since the key point in harmonic integration is mainly taking traces retaining $SL(2,C)$ singlets, we have then the following integration rules,

$$\int_{T^*P^1}(u^+)^{(m}(v^-)^{n)}(u^+)_{(k}(v^-)_{l)} = \frac{(-1)^n m!n!}{(m+n+1)!}\delta_{(\beta_1}^{(\alpha_1}\cdots\delta_{\beta_{k+l)}}^{\alpha_{m+n)}} \tag{9.51}$$

which is non zero only if $m = l$ and $n = k$. Here the harmonic monomials $(u^+)^{(m}(v^-)^{n)}$ stand for the short of the following upper index quantity,

$$(u^+)^{(m}(v^-)^{n)} \equiv u^{+(\alpha_1}...u^{+\alpha_m}v^{-\beta_1}...v^{-\beta_n)}, \tag{9.52}$$

and a similar relation for lower indices. As far as eq(9.51) it vanishes identically whenever

$$m + n \neq k + l. \tag{9.53}$$

In the case when this condition is fulfilled, we have non zero results. For instance, we have for $n = m = 0$, the normalized holomorphic volume of T^*P^1, that is

$$\int_{T^*P^1} = 1, \tag{9.54}$$

and for the case $n = k = 0$, the above harmonic integral reduces to

$$\int_{T^*P^1} (u^+)^m (v^-)_l = \frac{1}{(m+1)}\delta^{(\alpha_1}_{(\beta_1}\cdots\delta^{\alpha_m)}_{\beta_l)}, \tag{9.55}$$

whose non trivial leading term is

$$\int_{T^*P^1} u^{+\alpha}v^-_\beta = \frac{1}{2}\delta^\alpha_\beta. \tag{9.56}$$

On the real 2-sphere obtained from previous analysis by setting

$$v^- = u^-, \tag{9.57}$$

the above rules reduce to

$$\begin{aligned} u^+_\alpha u^+_{(\beta_1}\cdots u^+_{\beta_n}u^-_{\gamma_1}\cdots u^-_{\gamma_m)} &= u^+_{(\alpha}u^+_{\beta_1}\cdots u^-_{\gamma_m)} + \frac{m}{m+n+1}\varepsilon_{\alpha(\gamma_1}u^+_{\beta_1}\cdots u^+_{\beta_n}u^-_{\gamma_2}\cdots u^-_{\gamma_m)}, \\ u^-_\gamma u^+_{(\beta_1}..u^+_{\beta_n}u^-_{\gamma_1}..u^-_{\gamma_m)} &= u^-_{(\gamma}u^+_{\beta_1}..u^-_{\gamma_m)} - \frac{n}{m+n+1}\varepsilon_{\gamma(\beta_1}u^+_{\beta_2}..u^+_{\beta_n}u^-_{\gamma_1}..u^-_{\gamma_m)}. \end{aligned} \tag{9.58}$$

We also have the following harmonic integration rule on the unit 2-sphere,

$$\int_{S^2} du (u^+)^{(m}(u^-)^{n)}(u^+)_{(k}(u^-)_{l)} = \frac{(-1)^n m!n!}{(m+n+1)!}\delta^{(\alpha_1}_{(\beta_1}\cdots\delta^{\alpha_{m+n})}_{\beta_{k+l})}. \tag{9.59}$$

9.2.2 Delta function and harmonic distributions

An important tool in the harmonic integration on the real 2-sphere S^2, on which we focus our attention now on, is given by harmonic δ functions

$$\delta^{(q,-q)}(u_1, u_2) \tag{9.60}$$

extending usual one dimensional Dirac distribution. Like the usual ordinary δ function,

$$\delta(x_1 - x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp[ip(x_1 - x_2)]dp \tag{9.61}$$

diverging $x_1 = x_2$, this is a distribution with quite same features. It is defined as follows,

$$\int_{S^2} d^2u_2 \; \delta^{(q,-q)}(u_1^{\pm}, u_2^{\pm}) F^p(u_2^{\pm}) = F^q(u_1^{\pm})\delta^{pq}, \qquad q \in Z. \tag{9.62}$$

Viewed as a function on the 2-sphere, the $\delta^{(q,-q)}(u_1, u_2)$ functions have the generic harmonic expansion,

$$\delta^{(q,-q)}(u_1, u_2) = \sum_{n=0}^{\infty} (-1)^{n+q} \frac{(2n+q+1)!}{(n)!(n+q)!} (u_1^+)_{(n+q}(u_1^-)_{n)}(u_2^+)^{(n}(u_2^-)^{n+q)} \tag{9.63}$$

which is clearly divergent for

$$u_1^{\pm} = u_2^{\pm}. \tag{9.64}$$

The previous integration rule allows us to determine the coefficients $F^{(\alpha_1 \ldots \alpha_{q+n} \beta_1 \ldots \beta_n)}$ of the harmonic expansion (9.13). We have

$$F^{(\alpha_1 \ldots \alpha_{q+n} \; \beta_1 \ldots \beta_n)} = \frac{(-1)^{n+q}(2n+q+1)!}{(n+q)!n!} \int_{S^2} d^2u \; (u^+)^{(n}(u^-)^{n+q)} F^{(q)}(u) \tag{9.65}$$

Along with the generalized Dirac $\delta^{(q,-q)}$ distribution, there are others harmonic distributions which are particularly relevant in quantum field theory. A class of these harmonic distributions correspond to

$$\frac{1}{(u_1^+ u_2^+)^n} \tag{9.66}$$

and

$$\frac{1}{(v_1^- v_2^-)^n} \tag{9.67}$$

with singularity at $u_1^+ = u_2^+$ and $v_1^- = v_2^-$. For $\frac{1}{(u_1^+ u_2^+)^n}$, we have,

$$\frac{1}{(u_1^+ u_2^+)^n} = \frac{1}{n!} \sum_{k=0}^{\infty} (-1)^{n+k} \frac{(2k+n+1)!}{k!(k+1)!} \frac{n}{n+k} (u_1^+)_{(k}(v_1^-)_{k+n)}(u_2^+)^{(k}(v_2^-)^{k+n)}, \tag{9.68}$$

with $n > 0$. A similar harmonic distributions may be also written down for $(v_1^- v_2^-)^{-n}$. These distributions appear in the computation of Green functions. For recent explicit applications see for instance [59]-[61] and references therein.

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